2. (a) Show that every simple graph with at least two vertices has two vertices of same valency.

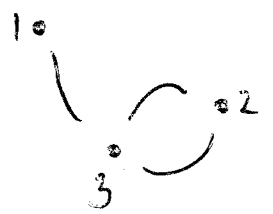
(Recall that a simple graph is a graph without multiple edges between pairs of vertices.)

(b) Give an example that this statement is false if you allow multiple edges.

(5+5)

(a) Suppose the graph has n vertices, as it is simple, each vertex can have valency at most $n-1$. So the set of possible valencies is $\{0, 1, \ldots, n-1\}$. Now suppose these values occur exactly once each. Then one vertex has valency 0, so it is not connected to any other vertex, and there can be no vertex with valency $n-1$, which is a contradiction.
So at least one of the admissible valencies must occur more than once.

(b)



3. Let G be a finite connected planar graph with V its set of vertices, E its set of edges, and F its set of faces.

   (a) Show that $2|E| \geq 3|F|$.

   (b) Show that $|E| \leq 3|V| - 6$.

   (c) Conclude that every planar graph with less than 12 vertices must have at least one vertex of valency less than 5.

(5+5+5)

(a) Each face is delineated by at least 3 edges, each edge delineates at most two faces.

(b) Euler's formula: $|V| - |E| + \underbrace{|F|}_{\leq \frac{2}{3}|E|} = 2$

$$\Rightarrow \frac{1}{3}|E| \leq |V| - 2$$

$$\Rightarrow |E| \leq 3|V| - 6$$

(c) From (b): Sum of valencies $\leq 6|V| - 12$

Suppose that every vertex has valency ≥ 5

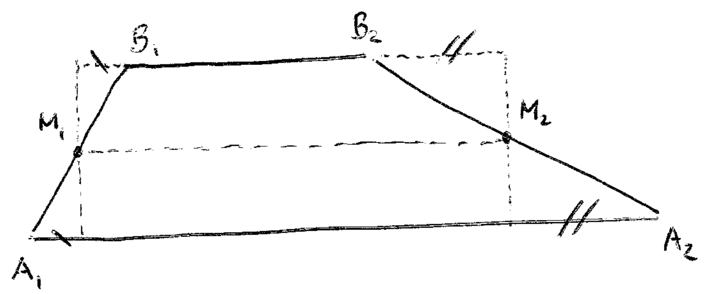
$$\Rightarrow 5|V| \leq 6|V| - 12$$

$$\Rightarrow |V| \geq 12$$

which contradicts the assumption $|V| < 12$.

4. Show that a quadrilateral (a polygon with four edges) is a trapezoid (a quadrilateral with two parallel edges) if and only if the length of the line segment joining the midpoints of a pair of opposite edges is equal to half the sum of the lengths of the other two edges. (10)

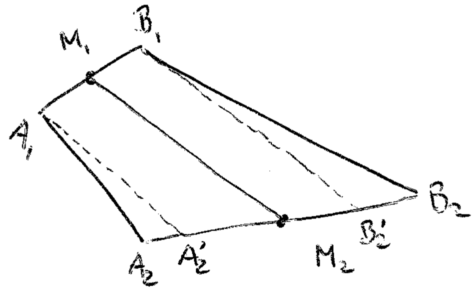
" $\Rightarrow$ ":



From figure, it is apparent that

$$2\,|\overline{M_1M_2}| = |\overline{A_1A_2}| + |\overline{B_1B_2}|$$

" $\Leftarrow$ ": For a general quadrilateral, shorten one of the sides until you obtain a parallelogram:



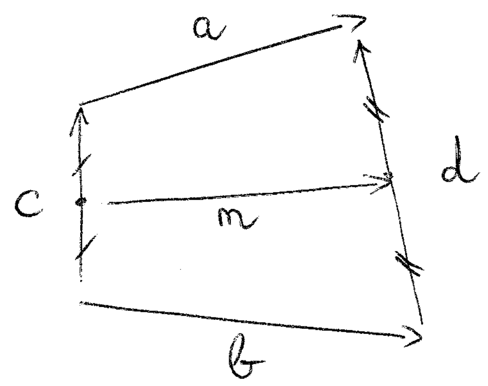
This does not affect $|\overline{M_1M_2}|$, and by the above argument

$$2|\overline{M_1M_2}| = |\overline{A_1A_2'}| + |\overline{B_1B_2'}|$$

Since $|\overline{A_1A_2}| > |\overline{A_1A_2'}|$ and $|\overline{B_1B_2}| > |\overline{B_1B_2'}|$ if not already a parallelogram, this contradicts the assumption.

Vector Algebra solution for Problem 4:

Consider a general quadrilateral:



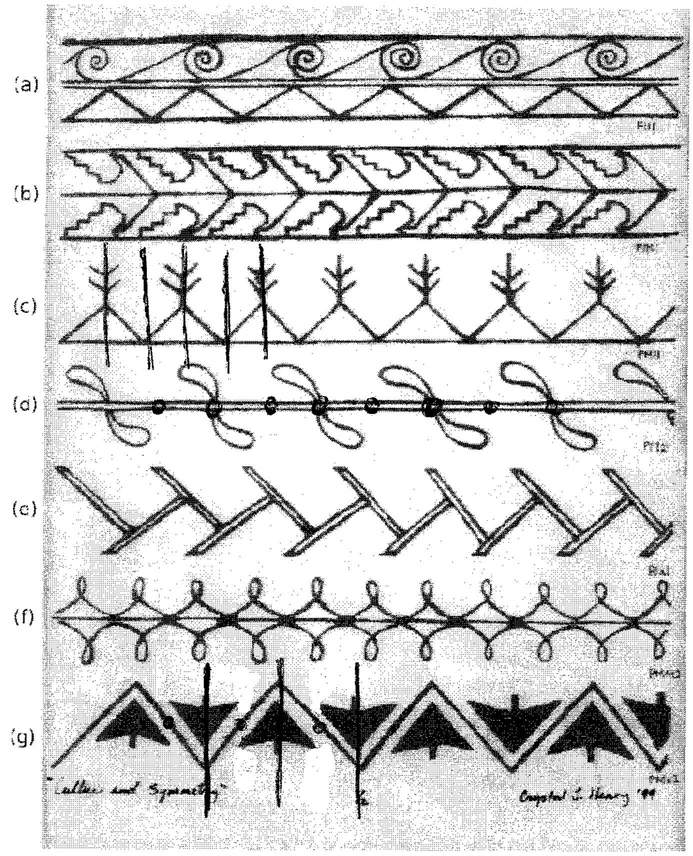
Then: $b = \frac{c}{2} + m - \frac{d}{2}$

$+)\quad a = -\frac{c}{2} + m + \frac{d}{2}$

$a + b = 2m$

Now $\|a\| + \|b\| = \|a+b\|$ if and only if a and b are parallel.

5. Describe the symmetry group for each of the following ornaments.



(a) only Π

(b) Π and $R_{\parallel}$

(c) Π and $R_{\perp}$

(d) Π and H_0

(e) $U_{1/2}$ (Π is subgroup)

(f) Π, $R_{\parallel}$, $R_{\perp}$, H_0

(g) $U_{1/2}$, H_0, $R_{\perp}$

Iroquois and Ojibwa border designs. From http://www.oswego.edu/~baloglou/103/crystal.html

(10)

6. Recall that the finite cyclic group of order n is

$$\begin{aligned} C_n &= \{\langle a \rangle : a^n = e\} \\ &= \{e, a, \ldots, a^{n-1}\} \end{aligned}$$

and that the symmetry group of an n-gon is the dihedral group

$$\begin{aligned} D_n &= \{\langle a, b \rangle : a^n = e, b^2 = e, ab = ba^{-1}\} \\ &= \{e, a, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}. \end{aligned}$$

Show that any finite subgroup of the motions of the plane (length-preserving transformations of the plane) is isomorphic to either C_n or D_n for some $n \geq 1$.

Note: This is a typical classification problem where you have to sift through all possibilities. Do not expect a short answer. Full credit will be given if the delineation of the problem is reasonably complete. Extra credit for a complete solution. (10)

Recall that all motions of the plane are translations, rotations, reflections, or glide reflections. We will cut down on the possibilities which can occur in a finite subgroup G of the group of motions.

① G cannot contain translations or glide reflections as any one would already generate an infinite subgroup of G.

② Suppose G contains a rotation R. Then the subgroup generated by R is either C_n for some $n \geq 1$ or is infinite, hence excluded.

③ Suppose G contains two non-trivial rotations about different centers. Then G also contains a translation, but this is excluded by ①.

(4) Suppose G contains two rotations about the same center. Then one generates a group C_n, the other a group C_m. Together they generate the group C_N, where N is the least common multiple of n and m. (This can be regarded, for example, as a consequence of the Chinese Remainder Theorem.)

(5) Suppose G contains line reflections. Then $G = D_n$ for some $n \geqslant 1$.

<u>Proof</u>: Let $\mathcal{S} = \{S_1, \dots, S_m\}$ denote all line reflections contained in G (which is finite!). From (3) and (4) it follows that the rotations in G are $\{R^0 = e, R^1, \dots, R^{n-1}\}$ for some $n \geqslant 0$. Let $S \in \mathcal{S}$. The composition of line reflection and rotation is again a line reflection, so

$$\tilde{\mathcal{S}} := \{S, SR, \dots, SR^{n-1}\}$$

is another set of line reflections, and $\tilde{\mathcal{S}} \subset \mathcal{S}$ by definition of $\mathcal{S}$. On the other hand, suppose there exists $S' \in \mathcal{S} \setminus \tilde{\mathcal{S}}$. Then SS' must be a rotation (as translation is ruled out already), whence $S' \in \tilde{\mathcal{S}}$, contradiction!

This proves $\mathcal{S} = \tilde{\mathcal{S}}$, so $G = D_n$.

This exhausts all possibilities.