Introduction to Partial Differential Equations

Homework 7

due May 12, 2011

1. (From Evans, p. 164, Question 13.) Let $u \in C(\mathbb{R} \times [0,T])$ for some $T > 0$ be a weak solution to the scalar conservation law

$$u_t + F(u)_x = 0 \qquad \text{in } \mathbb{R} \times (0,T)\,,$$
$$u = g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,.$$

Assume further that for any fixed $t \in [0,T]$, $u(\,\cdot\,,t)$ has compact support in $\mathbb{R}$, and that $F(0) = 0$. Show that

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} g(x)\,dx$$

for every $t \in [0,T]$.

2. (From Evans, p. 164, Question 14.) Compute explicitly the unique entropy solution of

$$u_t + u\,u_x = 0 \qquad \text{in } \mathbb{R} \times (0,\infty)\,,$$
$$u = g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,.$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x\,. \end{cases}$$

Draw the characteristic curves in an (x,t)-plot, being sure to document all qualitative changes in the solution for $t > 0$.

3. (From Evans, p. 163, Question 5.)

   (a) For $1 < r < \infty$, let $H\colon \mathbb{R}^n \to \mathbb{R}$ be defined as

   $$H(p) = \frac{1}{p}\,|p|^r\,.$$

   Show that its Legendre transform satisfies

   $$H^*(q) = \frac{1}{s}\,|q|^s\,, \qquad \text{where } \frac{1}{r} + \frac{1}{s} = 1\,.$$

(b) Let $H(p) = p^T Ap + b^T p$ where $A \in M(n \times n)$ is symmetric and positive definite, and $b \in \mathbb{R}^n$. Compute $H^*(q)$.

4. Consider the initial-value problem

$$\begin{aligned} u_t + |u_x|^2 = 0 \qquad &\text{in } \mathbb{R} \times (0, \infty)\,, \\ u = 0 \qquad &\text{on } \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

One non-trivial candidate solution which is Lipshitz and satisfies the equation almost everywhere, is given by Evans, p. 129:

$$u(x,t) = \begin{cases} 0 & \text{if } |x| \geq t\,, \\ x - t & \text{if } 0 \leq x \leq t\,, \\ -x - t & \text{if } -t \leq x \leq 0\,. \end{cases}$$

(a) Show that u is not semiconcave.

(b) Give another nontrivial solution which is Lipshitz and satisfies the equation almost everywhere.

(c) What is the solution given by the Hopf–Lax formula?

5. Fill in the gaps in the proof of Evans, p. 131, Lemma 4.

6. Evans, p. 164, Question 10.