1. (a) Solve the partial differential equation

$$t u_t + x u_x = 0\,,$$
$$u(x, 1) = g(x)\,,$$

where $u = u(x, t)$ using the method of characteristics. For $t \geq 1$.

(b) Draw the characteristic curves. Discuss how you could solve the equation in the exterior of the unit disk.

(5+5)

(a) Let $z(s) = u(x(s), t(s))$

$\Rightarrow \quad z'(s) = u_x(x(s), t(s))\, x'(s) + u_t(x(s), t(s))\, t'(s)$

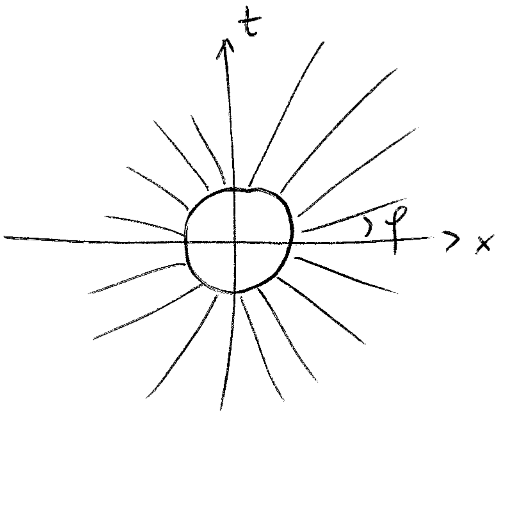
So we need $x'(s) = x(s)$ and $t'(s) = t(s)$

$\Rightarrow \quad x(s) = x_0 e^{s}$ and $t(s) = t_0 e^{s}$

to get $z'(s) = 0$

$\Rightarrow \quad \dfrac{x(s)}{t(s)} = \dfrac{x_0}{t_0}$, with $t_0 = 1$ we get $u(x,t) = g\left(\frac{x}{t}\right)$.

(b)



The characteristic curves are radial lines, so they can be parametrized by the angle $\varphi = \arg(x,y)$.

A unique solution in the exterior of the unit circle is then specified by some function $h(\varphi)$ of values of u on the unit circle in the (x,t)-plane.

2. Suppose u is a radial solution of the *Helmholtz equation*

$$u - \Delta u = 0 \qquad \text{in } \mathbb{R}^n.$$

Set $u(x) = v(r)$ with $r = |x|$. Show that v must satisfy the *modified Bessel equation*

$$v - (n-1)\frac{v'}{r} - v'' = 0.$$

(Do not attempt to solve it.) (10)

$$Du = v'(r)\frac{x}{r}$$

$$D \cdot Du = v''(r)\frac{x \cdot x}{r^2} + \underbrace{D \cdot x}_{=n} \frac{v'(r)}{r} - \underbrace{r^{-2}\, x \cdot \frac{x}{r}\, v'(r)}_{= \frac{v'(r)}{r}}$$

$$= v'' + (n-1)\frac{v'}{r}$$

$$\Rightarrow 0 = (1 - \Delta)u = v - \left(v'' + (n-1)\frac{v'}{r}\right) .$$

3. Let $U \subset \mathbb{R}^n$ be open, bounded, and connected. Let $u\colon U \to \mathbb{R}$ be a nonnegative harmonic function. Show that if $u(x) = 0$ for some $x \in U$, then $u = 0$ everywhere in U.
(10)

Solution 1: Use the maximum principle for $-u$; $u(x)=0$ is a maximum of $-u$ in the interior, hence $u \equiv 0$ everywhere.

Solution 2: The set $N = \{x \in U : u(x) = 0\}$ is relatively closed in U.

Suppose there is a point $x_0 \in U \cap \partial N$. Then $B(x_0, r) \subset U$ for sufficiently small $r>0$ and by the mean value formula,

$$0 = u(x_0) = \fint_{B(x_0,r)} u(x)\,dx .$$

Since $u \geq 0$, this can only be true if $u \equiv 0$ on $B(x_0, r)$

$\Rightarrow x_0 \notin \partial N$ ↯

We conclude that $N = U$, so $u \equiv 0$ on U.

4. Show that if the initial datum to the heat equation is even, the solution will be even for any fixed $t \geq 0$. (5)

Let $w(x,t) = u(-x,t)$, where $u_t = \Delta u$.

Then $w_t = \frac{\partial u}{\partial t}(-x,t) = \Delta u(-x,t) = \Delta(u(-x,t)) = \Delta w$,

so w solves the heat equation, too.

If u is even initially, $u=w$ initially.

By uniqueness, $u=w$ for $t>0$, so u even for $t>0$.

5. Suppose $u = u(x,t)$ is a smooth function on $\mathbb{R} \times [0,\infty)$ with compact support in the x-direction for every fixed $t \geq 0$. Suppose further that u solves Burgers' equation

$$u_t + F(u)_x = 0 \qquad \text{with} \qquad F(u) = \tfrac{1}{2}u^2 .$$

(a) Show that

$$M(t) = \int_{\mathbb{R}} u(x,t)\,dx \qquad \text{and} \qquad E(t) = \int_{\mathbb{R}} u(x,t)^2\,dx$$

are constants of the motion.

(b) Show that

$$\int_0^\infty \int_{\mathbb{R}} (u v_t + F(u)\,v_x)\,dx\,dt + \int_{\mathbb{R}} u(x,0)\,v(x,0)\,dx = 0 \qquad (*)$$

for every $v \in C^1(\mathbb{R} \times [0,\infty))$ with compact support.

(5+5)

(a) $$\frac{dM}{dt} = \int_{\mathbb{R}} u_t\,dx = -\int_{\mathbb{R}} F(u)_x\,dx = -\tfrac{1}{2}u^2\Big|_{-\infty}^{\infty} = 0$$

$$\frac{dE}{dt} = 2\int_{\mathbb{R}} u u_t\,dx = -2\int_{\mathbb{R}} u^2 u_x\,dx = -\frac{2}{3}\int_{\mathbb{R}} (u^3)_x\,dx = -\frac{2}{3}u^3\Big|_{-\infty}^{\infty} = 0$$

(b)

$$0 = \int_0^\infty \int_{\mathbb{R}} (u_t + F(u))\,v\,dx\,dt$$

$$= \int_{\mathbb{R}} uv\,dx\,\Big|_{t=0}^{t=\infty} - \int_0^\infty \int_{\mathbb{R}} u v_t\,dx\,dt$$

$$+ \int_0^\infty v\,F(u)\,dt\,\Big|_{x=-\infty}^{x=\infty} - \int_0^\infty \int_{\mathbb{R}} F(u)\,v_x\,dx\,dt$$

Of the boundary terms, only the term with $t=0$ may be nonzero, hence $(*)$.

6. *Note:* This problem continues Question 5 which should be attempted beforehand.

Suppose that u is an integral solution of Burgers' equation (i.e., it satisfies the condition stated in 5b) which is smooth everywhere except on a curve C. Suppose further that C has a smooth parametrization of the form $(s(t), t)$. Let u_l denote the left-hand limit of u on C and let u_r denote the right-hand limit of u on C.

(a) Show that, on C,

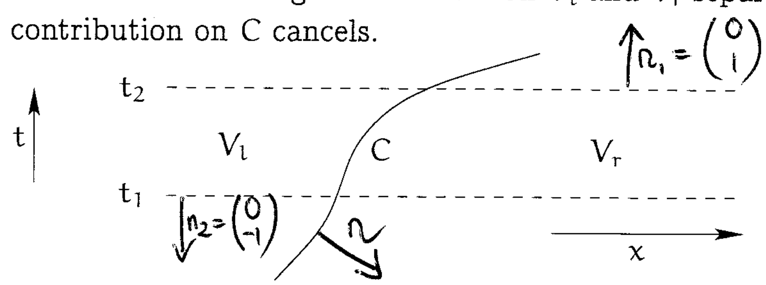
$$[[F(u)]] = \dot{s}\,[[u]]$$

where $[[F(u)]] = F(u_l) - F(u_r)$ and $[[u]] = u_l - u_r$ denote the jump in $F(u)$ and the jump in u across C, respectively.

Hint: Use the divergence theorem.

(b) Show that $M(t)$ is a constant of the motion.

Hint: Use the divergence theorem on V_l and V_r separately and notice that the contribution on C cancels.



(c) Suppose u satisfies the *entropy condition*

$$u(x+h, t) - u(x, t) \leq \frac{c}{t} h$$

for some constant $c > 0$. Show that this implies $u_l \geq u_r$.

(d) Show that, when u satisfies the entropy condition, $E(t)$ is a decreasing function of time.

Hint: Show that in the interior of V_l and V_r,

$$\tfrac{1}{2}\,\partial_t u^2 + \tfrac{1}{3}\,\partial_x u^3 = 0\,.$$

Now use the divergence theorem as in (b) and discuss the sign of the contribution on C.

(e) Give a physical interpretation of (d) vs. (b).

(5+5+5+5+5)

(a) Choose a test function v with compact support on $\mathbb{R}\times(0,\infty)$.

Then $(*)$ reads

$$\int_0^\infty \int_{\mathbb{R}} \left(u v_t + F(u) v_x \right) dx\, dt \qquad \text{Let } \underline{X} = \begin{pmatrix} x \\ t \end{pmatrix}, \quad D = \begin{pmatrix} \partial_x \\ \partial_t \end{pmatrix}$$

$$= \int_0^\infty \int_{x<s(t)} \begin{pmatrix} F(u) \\ u \end{pmatrix} \cdot D v \, d\underline{X}$$

$$+ \int_0^\infty \int_{s(t)<x} \begin{pmatrix} F(u) \\ u \end{pmatrix} \cdot D v \, d\underline{X}$$

$$= \int_0^\infty \int_{x<s(t)} \underbrace{D\cdot \begin{pmatrix} F(u) \\ u \end{pmatrix}}_{=0} v \, d\underline{X} + \int_0^\infty \int_{x>s(t)} \underbrace{D\cdot \begin{pmatrix} F(u) \\ u \end{pmatrix}}_{=0} v \, d\underline{X}$$

$$+ \int_C n\cdot \begin{pmatrix} F(u_l) \\ u_l \end{pmatrix} v \, dS - \int_C n\cdot \begin{pmatrix} F(u_r) \\ u_r \end{pmatrix} v \, dS \qquad (**)$$

where $\tilde{n} = \begin{pmatrix} 1 \\ -\dot{s} \end{pmatrix}$ and $n = \dfrac{\tilde{n}}{|\tilde{n}|}$

Since $(**)$ holds for arbitrary v, we find that on C,

$$n \cdot \begin{pmatrix} [\![F(u)]\!] \\ [\![u]\!] \end{pmatrix} = 0 \quad \Rightarrow \quad \dot{s}\,[\![u]\!] = [\![F(u)]\!].$$

(This is the Rankine-Hugoniot shock condition.)

(b)

$$0 = \int_{V_\ell} \underline{D} \cdot \begin{pmatrix} F(u) \\ u \end{pmatrix} d\underline{X} = \int_{\partial V_\ell} n \cdot \begin{pmatrix} F(u) \\ u \end{pmatrix} dS$$

$$= \int_{\{t=t_1,\, x<S(t_1)\}} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} F(u) \\ u \end{pmatrix} dS + \int_{C} \begin{pmatrix} 1 \\ -\dot{S} \end{pmatrix} \cdot \begin{pmatrix} F(u_\ell) \\ u_\ell \end{pmatrix} \frac{dS}{|n|}$$

$$+ \int_{\{t=t_2,\, x<S(t_2)\}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} F(u) \\ u \end{pmatrix} dS$$

$$= -\int_{-\infty}^{S(t_1)} u(x, t_1)\, dx + \int_{-\infty}^{S(t_2)} u(x, t_2)\, dx + \int_{C} n \cdot \begin{pmatrix} F(u_\ell) \\ u_\ell \end{pmatrix} dS$$

Similarly:

$$0 = -\int_{S(t_1)}^{\infty} u(x,t_1)\, dx + \int_{S(t_2)}^{\infty} u(x,t_2)\, dx - \int_{C} n \cdot \begin{pmatrix} F(u_r) \\ u_r \end{pmatrix} dS$$

Adding both relations gives

$$0 = -\underbrace{\int_{-\infty}^{\infty} u(x,t_1)\, dx}_{= M(t_1)} + \underbrace{\int_{-\infty}^{\infty} u(x,t_2)\, dx}_{= M(t_2)} + \underbrace{\int_{C} n \cdot \begin{pmatrix} [\![F(u)]\!] \\ [\![u]\!] \end{pmatrix} dS}_{=0 \text{ by (a)}}$$

(c) The entropy condition directly implies

$$u(x+h,t) - u(x-h,t) \leq \frac{C}{t} 2h$$

Fix t and set $x = s(t)$. Then, as $h \to 0$,

$$u(x+h,t) - u(x-h,t) \to u_r - u_l$$

$$\frac{C}{t} 2h \qquad \to 0$$

$\Rightarrow \quad u_l \geq u_r$.

(d) Inside of V_l and V_r, the integral solution satisfies the strong form of Burgers' equation, $u_t + u u_x = 0$.

Multiply with u so that $\underbrace{u u_t}_{=\frac{1}{2}(u^2)_t} + \underbrace{u^2 u_x}_{=\frac{1}{3}(u^3)_x} = 0$

Then

$$0 = \int_{V_l} D \cdot \begin{pmatrix} \frac{1}{3}u^3 \\ \frac{1}{2}u^2 \end{pmatrix} d\underline{X} = \ldots = -\int_{-\infty}^{s(t_1)} \tfrac{1}{2} u^2(x,t_1)\, dx + \int_{s(t_2)}^{\infty} \tfrac{1}{2} u^2(x,t_2)\, dx$$

$$+ \int_{G} n \cdot \begin{pmatrix} \frac{1}{3} u_l^3 \\ \frac{1}{2} u_l^2 \end{pmatrix} dS$$

and similarly on V_r.

$$\Rightarrow 0 = \underbrace{-\frac{1}{2}\int_{-\infty}^{\infty} u^2(x,t_1)\, dx}_{=E(t_1)} + \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} u^2(x,t_2)\, dx}_{=E(t_2)} + \int_{G} n \cdot \begin{pmatrix} \frac{1}{3}[\![u^3]\!] \\ \frac{1}{2}[\![u^2]\!] \end{pmatrix} dS$$

where $$n \cdot \begin{pmatrix} \frac{1}{3}[\![U^3]\!] \\ \frac{1}{2}[\![U^2]\!] \end{pmatrix} = |n|^{-1}\left[\frac{1}{3}U_\ell^3 - \frac{1}{3}U_r^3 - \dot{S}\,\underbrace{\left(\tfrac{1}{2}U_\ell^2 - \tfrac{1}{2}U_r^2\right)}_{=\frac{1}{2}(U_\ell - U_r)(U_\ell + U_r)}\right]$$

$$\underbrace{\dot{S}\left(\tfrac{1}{2}U_\ell^2 - \tfrac{1}{2}U_r^2\right)}_{} \overset{(a)}{=} \frac{1}{4}\left(U_\ell^2 - U_r^2\right)\left(U_\ell + U_r\right)$$

$$= \frac{1}{|n|}\left[\frac{1}{3}U_\ell^3 - \frac{1}{3}U_r^3 - \frac{1}{4}U_\ell^3 + \frac{1}{4}U_r^3 - \frac{1}{4}U_\ell^2 U_r + \frac{1}{4}U_\ell U_r^2\right]$$

$$= \frac{1}{12\,|n|}\left[U_\ell^3 - 3U_\ell^2 U_r + 3U_\ell U_r^2 - U_r^3\right]$$

$$= \frac{1}{12\,|n|}\left(U_\ell - U_r\right)^3 \overset{(c)}{\geq} 0$$

Hence, if $U_\ell > U_r$ somewhere,

$$E(t_2) < E(t_1)$$

(e) If U denotes a velocity, M may be interpreted as momentum of a unit density medium; E as energy.

Then momentum is preserved even when shocks are present, but energy is dissipated in a shock.

This is analogous to inelastic collisions in classical mechanics.