1. Let

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 3 & 2 & 7 \\ 2 & 1 & 4 \end{pmatrix}.$$

Find a basis for Ker A and Range A. (10)

We start by bringing A into row-echelon form:

$$\begin{pmatrix} 2 & 0 & 2 \\ 3 & 2 & 7 \\ 2 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\operatorname{Ker} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$. Moreover, the pivots are in columns 1 and 2, so that

$$\operatorname{Range} A = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

2. Let P_2 denote the vector space of polynomials up to degree 2 endowed with inner product

$$\langle p, q\rangle = \frac{1}{2}\int_{-1}^{1} p(x)\, q(x)\, dx\,.$$

and basis $B = \{1, x, x^2\}$.

(a) Find an orthogonal basis $E = \{e_1, e_2, e_3\}$ for P_2.
Note: To simplify computations, have a careful look at which vectors in B are already orthogonal. Normalization is not required.

(b) Find the matrix S representing the change of basis from $\overset{E}{\cancel{B}}$ to $\overset{B}{\cancel{E}}$ and compute S^{-1}.

(c) Define a linear map $L\colon V \to V$ by

$$(Lp)(x) = x\,p'(x)\,.$$

Find the matrix representation of L with respect to the basis B.

(d) Find the matrix representation of L with respect to the basis E.

(5+5+5+5)

(a) Note that $b_2 = x$ is orthogonal to both $b_1 = 1$ and $b_3 = x^2$.
(You can verify this by direct computation, or by noting that even functions are orthogonal to odd functions with respect to this scalar product.)

So we only need to use Gram-Schmidt on the subspace spanned by b_1 and b_3:

$$e_1 = 1 \qquad \Rightarrow \quad \|e_1\|^2 = \langle e_1, e_1\rangle = 1$$

$$e_2 = x$$

$$e_3 = b_3 - \langle b_3, e_1\rangle\, e_1$$

$$= x^2 - \frac{1}{2}\int_{-1}^{1} x^2\, dx \cdot 1 \quad = \quad x^2 - \frac{1}{3}$$

(b) We read off the coefficients of S from (a):

$$S = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By inspection,

$$S^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$L b_1 = x \cdot 0 = 0$$

$$L b_2 = x \cdot 1 = x$$

$$L b_3 = x \cdot 2x = 2x^2$$

$$\Rightarrow M_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(d)

$$\begin{tikzcd}
(V,\mathcal{B}) \arrow[r, "M_{\mathcal{B}}"] & (V,\mathcal{B}) \\
(V,E) \arrow[u, "S"'] \arrow[r, "M_E"'] \arrow[r, bend left=80, dashed] & (V,E) \arrow[u, "S"']
\end{tikzcd}$$

So we read off the diagram

$$M_E = S^{-1} M_{\mathcal{B}} S$$

$$= S^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3. (E) Let W be a nonempty proper subspace of $\mathbb{R}^n$ and let P be the orthogonal projector onto W. What are the eigenvalues of P? (Explain!) (8)

(A) Let V be the vector space of continuous functions on $\mathbb{R}$. Define a linear operator M by

$$(Mf)(x) = m(x)\, f(x)$$

where $m \in V$ is fixed. Under which conditions on m does M have eigenvalues? How are these eigenvalues characterized? (10)

(E) If $v \in W$, then $Pv = v$. Therefore, v is an eigenvector with eigenvalue 1.

If $v \in W^\perp = \{u \in \mathbb{R}^n : u^T w = 0 \text{ for all } w \in W\}$, then $Pv = 0$. Therefore, v is an eigenvector with eigenvalue 0.

Since $\dim W + \dim W^\perp = n$, we can be sure we found all eigenvalues. By assumption, $0 < \dim W < n$, so both eigenvalues 0 and 1 must occur.

(A) If f is an eigenvector, it must satisfy

$$m(x)\, f(x) = \lambda f(x)$$

$$\Rightarrow (m(x) - \lambda)\, f(x) = 0$$

So either $m(x) = \lambda$ or $f(x) \stackrel{!}{=} 0$. Since f cannot be zero for all $x \in \mathbb{R}$ (an eigenvector must not be the zero vector), there must be some $x_0 \in \mathbb{R}$ with $f(x_0) \neq 0$. Since f is cont., $f(x) \neq 0$ for all x in an interval containing x_0. Hence, $m(x) = \lambda$ on an interval

3.dd. So, whenever M has an eigenvalue, m must be constant on some interval. Vice versa, if m is constant on some interval, any f which is zero outside this interval is an eigenvector and this constant is the eigenvalue.

4. Given a 2π-periodic function with complex Fourier coefficients c_k, set $g(x) = f(x+a)$. Show that g has complex Fourier coefficients $e^{ika}\,c_k$. (10)

$$g_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\, g(x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\, f(x+a)\, dx$$

$$= \frac{1}{2\pi}\int_a^{2\pi+a} e^{-ik(y-a)}\, f(y)\, dy$$

$$= e^{ika}\,\frac{1}{2\pi}\int_a^{2\pi+a} e^{-iky}\, f(y)\, dy$$

$$= e^{ika}\,\frac{1}{2\pi}\int_0^{2\pi} e^{-iky}\, f(y)\, dy \quad \text{(by periodicity)}$$

$$= e^{ika}\, c_k .$$

5. Compute

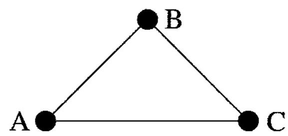
$$\int_{-\infty}^{\infty} x^2\,\delta(x^3)\,dx\,.$$

(10)

Let $u = x^3 \Rightarrow du = 3x^2\,dx$

$$\Rightarrow \int_{-\infty}^{\infty} x^2\,\delta(x^3)\,dx = \int_{-\infty}^{\infty} \delta(u)\,\frac{du}{3} = \frac{1}{3}\,.$$

6. Consider the following communication network:



Assume the links between stations A, B, and C may fail independently of each other with failure probabilities $P(AC) = \frac{1}{2}$, $P(AB) = \frac{1}{3}$, and $P(BC) = \frac{1}{4}$, respectively.

(a) What is the probability that there is a path in the network from station A to C?

(b) You are probing the network by sending a signal from station A and find that you receive the signal at station C. What is the probability that the direct link AC is working?

(7+8)

(a) $P(A \text{ connected to } C) = P(\overline{AC} \cup (\overline{AB} \cap \overline{BC}))$

$$= P(\overline{AC}) + P(\overline{AB} \cap \overline{BC}) - P(\overline{AC} \cap \overline{AB} \cap \overline{BC})$$

$$= P(\overline{AC}) + P(\overline{AB})\, P(\overline{BC}) - P(\overline{AC})\, P(\overline{AB})\, P(\overline{BC})$$

$$= \frac{1}{2} + \frac{2}{3}\,\frac{3}{4} - \frac{1}{2}\,\frac{2}{3}\,\frac{3}{4}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$

(b) Let S denote the event of receiving the signal at C, so $P(S) = \frac{3}{4}$.

$$P(\overline{AC} \mid S) = \frac{P(S \mid \overline{AC})\, P(\overline{AC})}{P(S)}$$

$$= \frac{1 \cdot \frac{1}{2}}{\frac{3}{4}}$$

$$= \frac{2}{3}$$

7. The exponential distribution is a continuous probability distribution with probability distribution function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

(a) Show that f is indeed a probability distribution function by checking that it is properly normalized.

(b) Derive its moment generating function $M(t) = E[e^{tX}]$.

(c) Find its mean and variance.

(5+5+5)

(a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx = \frac{\lambda}{-\lambda} e^{-\lambda x} \Big|_{0}^{\infty} = 1$

(provided $\lambda > 0$.)

(b) $M(t) = \int_{-\infty}^{\infty} f(x)\, e^{tx}\,dx = \lambda \int_{0}^{\infty} e^{(t-\lambda)x}\,dx$

$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_{0}^{\infty} = \frac{\lambda}{\lambda - t}$ for $t < \lambda$

(c) $E[X] = M'(0) = \lambda (-1)(-1)(\lambda - t)^{-2} \Big|_{t=0} = \frac{1}{\lambda}$

$Var[X] = M''(0) - M'(0)^2$

$= \lambda(-1)(-2)(\lambda - t)^{-3} \Big|_{t=0} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

8. There are two types of students. Students of type A report sick only when they are actually sick. Students of type B report sick with probability 1 whenever there is an exam. It is known that the sickness rate of the general working population is 5%. On every exam, 10% of students report sick.

   (a) What is the probability that a randomly selected student is of type B? (5)

   (b) (E) What is the probability that a student who reports sick on the day of the exam is of type B? (4)

   (A) A student reports sick on the midterm and on the final. What is the probability that the student is of type B? (5)

(a) Let S denote the event that a student reports sick at an exam.

$$\underbrace{P(S)}_{=\frac{1}{10}} = \underbrace{P(S|A)}_{=\frac{1}{20}}\, \underbrace{P(A)}_{=1-P(B)} + \underbrace{P(S|B)}_{=1} P(B)$$

$$\Rightarrow \frac{1}{10} - \frac{1}{20} = P(B)\left(1 - \frac{1}{20}\right)$$

$$\Rightarrow P(B) = \frac{\frac{1}{20}}{\frac{19}{20}} = \frac{1}{19}$$

(b) (E) $$P(B|S) = \frac{P(S|B)\cdot P(B)}{P(S)} = \frac{1\cdot\frac{1}{19}}{\frac{1}{10}} = \frac{10}{19} \approx \frac{1}{2}$$

(A) Let S_1 be the event that a student reports sick on the midterm, S_2 be the event that a student reports sick on the final

$$P(B|S_1\cap S_2) = \frac{P(S_1\cap S_2|B)\ P(B)}{P(S_1\cap S_2)} = \frac{P(S_1\cap S_2|B)\ P(B)}{P(S_1\cap S_2|A)\,P(A) + P(S_1\cap S_2|B)\,P(B)}$$

$$= \frac{1\cdot\frac{1}{19}}{\frac{1}{20}\,\frac{1}{20}\,\frac{18}{19} + 1\cdot\frac{1}{19}} = \frac{1}{\frac{9}{200}+1} = \frac{200}{209} \approx 0.96$$

9. Let $X_1, \ldots, X_N$ be identically and independently distributed random variables with $E[X_i] = \mu$ and $Var[X_i] = \sigma^2$.

(E) Show that

$$E\left[\sum_{i=1}^{N} \frac{X_i}{N}\right] = \mu \quad \text{and} \quad E\left[\sum_{i=1}^{N} \frac{(X_i-\mu)^2}{N}\right] = \sigma^2.$$

(8)

(A) Define the sample mean

$$\mu_{\text{sample}} = \sum_{i=1}^{N} \frac{X_i}{N}$$

and the sample variance

$$\sigma^2_{\text{sample}} = \sum_{i=1}^{N} \frac{(X_i - \mu_{\text{sample}})^2}{N-1}.$$

Show that $E[\sigma^2_{\text{sample}}] = \sigma^2$. (10)

(E) $E\left[\frac{X_1+\ldots+X_N}{N}\right] = \frac{1}{N}\left(E[X_1]+\ldots+E[X_N]\right) = \frac{1}{N} N\mu = \mu$

$$E\left[\sum_{i=1}^{N}\frac{(X_i-\mu)^2}{N}\right] = \frac{1}{N}\sum_{i=1}^{N}\underbrace{E[(X_i-\mu)^2]}_{=Var[X_i]=\sigma^2} = \frac{1}{N}N\sigma^2 = \sigma^2$$

(A) By symmetry of the expression under index permutation,

$$E[\sigma^2_{sample}] = \frac{N}{N-1}E\left[\left(X_1 - \frac{X_1+\ldots+X_N}{N}\right)^2\right] = \frac{N}{N-1}E\left[\left(X_1\cdot\frac{N-1}{N} - \underbrace{\left(\frac{X_2}{N}+\ldots+\frac{X_N}{N}\right)}_{=:Z}\right)^2\right]$$

But Z is a random variable, independent of X_1, with mean $\frac{N-1}{N}\mu$ and variance $\frac{N-1}{N^2}\sigma^2$.

$$\Rightarrow E[\sigma^2_{sample}] = \frac{N}{N-1}E\left[\left(\frac{N-1}{N}\right)^2 X_1^2 - 2\frac{N-1}{N}X_1 Z + Z^2\right]$$

$$= \frac{N}{N-1}\left(\left(\frac{N-1}{N}\right)^2(\sigma^2+\mu^2) - 2\left(\frac{N-1}{N}\right)^2\mu^2 + \frac{N-1}{N^2}\sigma^2 + \left(\frac{N-1}{N}\right)^2\mu^2\right) = \frac{N-1}{N}\sigma^2 + \frac{1}{N}\sigma^2 = \sigma^2$$