Engineering and Science Mathematics 2B

Final Exam

May 23, 2013

1. Let
$$A = \begin{pmatrix} 2 & 0 & 2 \\ 3 & 2 & 7 \\ 2 & 1 & 4 \end{pmatrix} .$$
Find a basis for $\operatorname{Ker} A$ and $\operatorname{Range} A$. (10)

2. Let P_2 denote the vector space of polynomials up to degree 2 endowed with inner product
$$\langle p, q \rangle = \frac{1}{2} \int_{-1}^{1} p(x)\, q(x)\, \mathrm{d}x .$$
and basis $B = \{1, x, x^2\}$.
   (a) Find an orthogonal basis $E = \{e_1, e_2, e_3\}$ for P_2.
   Note: To simplify computations, have a careful look at which vectors in B are already orthogonal. Normalization is not required.
   (b) Find the matrix S representing the change of basis from E to B and compute S^{-1}.
   (c) Define a linear map $L \colon V \to V$ by
   $$(Lp)(x) = x\, p'(x) .$$
   Find the matrix representation of L with respect to the basis B.
   (d) Find the matrix representation of L with respect to the basis E.

(5+5+5+5)

3. (E) Let W be a nonempty proper subspace of $\mathbb{R}^n$ and let P be the orthogonal projector onto W. What are the eigenvalues of P? (Explain!) (8)
   (A) Let V be the vector space of continuous functions on $\mathbb{R}$. Define a linear operator M by
   $$(Mf)(x) = m(x)\, f(x)$$
   where $m \in V$ is fixed. Under which conditions on m does M have eigenvalues? How are these eigenvalues characterized? (10)

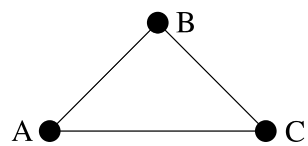
4. Given a 2π-periodic function with complex Fourier coefficients c_k, set $g(x) = f(x+a)$. Show that g has complex Fourier coefficients $\mathrm{e}^{ika}\, c_k$. (10)

5. Compute

$$\int_{-\infty}^{\infty} x^2\, \delta(x^3)\, \mathrm{d}x\,.$$

(10)

6. Consider the following communication network:



Assume the links between stations A, B, and C may fail independently of each other with failure probabilities $P(AC) = \frac{1}{2}$, $P(AB) = \frac{1}{3}$, and $P(BC) = \frac{1}{4}$, respectively.

(a) What is the probability that there is a path in the network from station A to C?

(b) You are probing the network by sending a signal from station A and find that you receive the signal at station C. What is the probability that the direct link AC is working?

(7+8)

7. The *exponential distribution* with parameter $\lambda > 0$ is a continuous probability distribution with probability distribution function

$$f(x) = \begin{cases} \lambda\, \mathrm{e}^{-\lambda x} & \text{for } x \geq 0\,, \\ 0 & \text{for } x < 0\,. \end{cases}$$

(a) Show that f is indeed a probability distribution function by checking that it is properly normalized.

(b) Derive its moment generating function $M(t) = E[\mathrm{e}^{tX}]$.

(c) Find its mean and variance.

(5+5+5)

8. There are two types of students. Students of type A report sick only when they are actually sick. Students of type B report sick with probability 1 whenever there is an exam. It is known that the sickness rate of the general working population is 5%. On every exam, 10% of students report sick.

(a) What is the probability that a randomly selected student is of type B? (5)

(b) (E) What is the probability that a student who reports sick on the day of the exam is of type B? (4)

(A) A student reports sick on the midterm and on the final. What is the probability that the student is of type B? (5)

9. Let $X_1, \ldots, X_N$ be identically and independently distributed random variables with $E[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2$.

(E) Show that

$$E\left[\sum_{i=1}^{N} \frac{X_i}{N}\right] = \mu \qquad \text{and} \qquad E\left[\sum_{i=1}^{N} \frac{(X_i - \mu)^2}{N}\right] = \sigma^2 \, .$$

(8)

(A) Define the *sample mean*

$$\mu_{\text{sample}} = \sum_{i=1}^{N} \frac{X_i}{N}$$

and the *sample variance*

$$\sigma^2_{\text{sample}} = \sum_{i=1}^{N} \frac{(X_i - \mu_{\text{sample}})^2}{N-1} \, .$$

Show that $E[\sigma^2_{\text{sample}}] = \sigma^2$. (10)