Engineering and Science Mathematics 2B

Homework 5

due March 20, 2013, before 12:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Using the Gram–Schmidt procedure, construct an orthonormal set of vectors from

$$\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} , \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} , \quad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} , \quad \boldsymbol{v}_4 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} .$$

2. (E) Using the Gram–Schmidt procedure, find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} , \quad \boldsymbol{v}_2 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} .$$

   (A) Let $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ be an orthonormal set of vectors in a vector space V with an inner product. Bessel's inequality (see p. 351 in the book) states that for every vector $\boldsymbol{x} \in V$,

$$\|\boldsymbol{x}\|^2 \geq \sum_{j=1}^{m} |\langle \boldsymbol{x}, \boldsymbol{w}_j \rangle|^2 .$$

   Prove that Bessel's inequality becomes an equality if and only if

$$\boldsymbol{x} \in \operatorname{Span}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\} .$$

3. A matrix A is called unitary if $A^H = A^{-1}$. Explain why this is the same as saying that the columns of A are orthonormal, or that the rows of A are orthonormal.

4. (E) Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   and show that the eigenvectors are orthogonal.

(A) Let

$$A = \begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

where α and β are non-zero complex numbers. Find its eigenvalues and eigenvectors. Find conditions for (a) the eigenvalues to be real and (b) the eigenvectors to be orthogonal. Show that the conditions are jointly satisfied if and only if A is Hermitian.

5. Let P_2 be the vector space of polynomials with real coefficients of degree less or equal to 2, and consider the inner product

$$\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, \mathrm{d}x \,.$$

(E) Apply the Gram–Schmidt procedure to the basis $\{1, x, x^2\}$.

(A) In addition to (E), explain *briefly* why $\langle \cdot \, , \, \cdot \rangle$ is an inner product.

6. Consider the vector space of continuous complex-valued functions on $[0, 2\pi]$ with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(x)\, g(x)\, \mathrm{d}x \,.$$

Show that the functions $f(x) = \mathrm{e}^{\mathrm{i}kx}$ and $g(x) = \mathrm{e}^{\mathrm{i}jx}$ are orthogonal when $k \neq j$.