Introductory Partial Differential Equations

Final Exam

May 30, 2013

1. Recall that the solution to the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)\,,$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy\,,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|z|^2}{4t}}\,.$$

(a) Show that

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \le \frac{C}{t^{n/2}}$$

provided that

$$\|g\|_{L^1} = \int_{\mathbb{R}^n} |g(x)|\, \mathrm{d}x < \infty\,.$$

(b) Suppose w solves the Poisson equation

$$-\Delta w = f \qquad \text{in } \mathbb{R}^n$$

where f is smooth and compactly supported. Show that the solution to the inhomogeneous heat equation

$$u_t - \Delta u = f \quad \text{in } \mathbb{R}^n \times (0, \infty)\,,$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}$$

tends to w as $t \to \infty$, i.e., that

$$\lim_{t \to \infty} u(x,t) = w(x)$$

for every $x \in \mathbb{R}^n$.

(10+10)

2. Let U be the open unit ball in $\mathbb{R}^n$. Suppose that $u \in C_1^2(\overline{U}_T)$ solves the heat equation

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } U_T\,, \\ u &= g && \text{on } U \times \{t=0\}\,, \\ u &= 0 && \text{on } \partial U \times [0,T]\,, \end{aligned} \tag{H}$$

where $U_T = U \times (0,T]$ and $g = 0$ on ∂U.

(a) Show that a radial solution $u(x,t) \equiv v(r,t)$ with $r = |x|$ satisfies

$$\begin{aligned} v_t &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial v}{\partial r} \right) && \text{in } (0,1) \times (0,T]\,, \\ v &= g && \text{on } (0,1) \times \{t=0\}\,, \\ v &= 0 && \text{on } \{r=1\} \times [0,T]\,. \end{aligned} \tag{R}$$

(b) Show that if $v \in C_1^2([0,1] \times [0,T])$ solves (R), it is the unique such solution. (In particular, show that a boundary condition at $r = 0$ is not required. Why would you expect this to be so?)

(c) When setting up a numerical discretization scheme for (R), a boundary condition at $r = 0$ might be necessary. What condition could you suggest?

(10+5+5)

3. Consider the linear transport equation

$$\begin{aligned} u_t + b\, u_x &= 0 && \text{in } \mathbb{R} \times (0,\infty)\,, \\ u &= g && \text{on } \mathbb{R} \times \{t=0\}\,. \end{aligned}$$

We say that $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is an integral solution of the transport equation provided

$$\int_0^\infty \int_{\mathbb{R}} u\,(v_t + b\,v_x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} g(x)\,v(x,0)\,\mathrm{d}x = 0$$

for all $v \in C_c^\infty(\mathbb{R} \times [0,\infty))$.

(a) Suppose that $g \in C(\mathbb{R})$ and that $u \in C^1(\mathbb{R} \times [0,\infty))$ is an integral solution. Show that u solves the transport equation in the classical sense.

(b) Now suppose that $g(x)$ is bounded and smooth except at some $a \in \mathbb{R}$ where it has a jump discontinuity. Show that $u(x,t) = g(x - bt)$ is an integral solution.

(10+10)

4. Find the entropy solution for Burgers' equation

$$\begin{aligned} u_t + u\,u_x &= 0 \qquad \text{in } \mathbb{R} \times (0, \infty)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

with initial data

$$g(x) = \begin{cases} 1 - x & \text{for } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) on the interval $t \in (0, 1)$;

(b) for $t \geq 1$.

Hint: Recall that the Rankine–Hugoniot shock condition for a conservation law of the form $u_t + F(u)_x = 0$ states that the solution at an isolated shock curve parameterized by $(s(t), t)$ satisfies $\dot{s}\,[u] = [F(u)]$, the brackets denoting the jump of the enclosed quantity across the shock. (10+10)

5. Let $u \in C(\mathbb{R}^3, \mathbb{R}^3)$ be a vector field with

$$|u(x)| \leq \frac{1}{1 + |x|^3}\,.$$

Show that

$$\int_{\mathbb{R}^3} \operatorname{div} u \, \mathrm{d}x = 0\,.$$

(10)

6. Let $u(x, t)$ be a smooth solution to the *Korteweg–de Vries equation*

$$u_t - 6\,u\,u_x + u_{xxx} = 0$$

on $\mathbb{R} \times (0, \infty)$ such that for every fixed $t \geq 0$, $u(x, t)$ and all its derivatives converge to zero as $|x| \to \infty$.

Show that

$$E(t) = \int_{\mathbb{R}} u(x, t)^2 \, \mathrm{d}x$$

remains constant in time. (10)