1. Let $g = C^1(\mathbb{R})$. Solve the partial differential equation

$$(x+t)(u_x + u_t) = 0 \qquad \text{in } \mathbb{R} \times (0,\infty), \qquad (*)$$
$$u = g \qquad \text{on } \mathbb{R} \times \{t = 0\}.$$

(10)

To find the characteristics, set

$$z(s) = u(x(s), t(s))$$

$$\Rightarrow z'(s) = u_x(x(s), t(s))\, x'(s) + u_t(x(s), t(s))\, t'(s)$$

To match with $(*)$, we need $x' = t'$

$$\Rightarrow dx = dt$$

$$\Rightarrow \int_{x_0}^{x} dx = \int_0^t dt$$

$$\Rightarrow x - t = x_0$$

$$\Rightarrow u(x,t) = u(x_0, 0) = g(x-t)$$

2. Suppose that a radial function $u = u(|x|)$ is harmonic on $B(0,1) \subset \mathbb{R}^n$. Show that $u \equiv \text{const}$. (10)

Since u is harmonic,

$$u(0) = \fint_{\partial B(0,r)} u(x)\, dS(x) \qquad \text{for } 0<r<1$$

$$= u(r) \fint_{\partial B(0,r)} dS \quad = u(r)$$

So $u = \text{const}$ on $B(0,1)$. (And extends continuously to $\overline{B(0,1)}$.)

3. Suppose that $U \subset \mathbb{R}^n$ is open, connected, and bounded with smooth boundary. Suppose further that $u \in C^2(\overline{U})$ solves the *Neumann problem* for the Poisson equation

$$\begin{aligned} -\Delta u = f \qquad & \text{in } U\,, \\ \nu \cdot Du = g \qquad & \text{on } \partial U \end{aligned}$$

for some $f \in C(\overline{U})$ and $g \in C(\partial U)$, where ν denotes the outer unit normal on ∂U. Show that any other solution differs from u by only a constant. (10)

Suppose v is such other solution. Then $w = u - v$ satisfies

$$-\Delta w = 0 \quad \text{in } \bar{U}$$

$$\nu \cdot Dw = 0 \quad \text{on } \partial \bar{U}$$

Now, since $\Delta w = 0$,

$$0 = -\int_{\bar{U}} w \, \Delta w \, dx$$

$$= -\int_{\partial U} w \underbrace{\nu \cdot Dw}_{=0} \, dS + \int_{\bar{U}} |Dw|^2 \, dx$$

Since $|Dw|^2$ is non-negative, the last integral can only vanish if $Dw = 0$ pointwise. This means that $w = \text{const}$ on each connected component of $\bar{U}$.

4. Suppose that $u \in C^2_1(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ solves the heat equation

$$u_t - \Delta u = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty)$$

and is a Gaussian at the initial time, i.e.,

$$u(x,0) = a\, e^{-b|x|^2}$$

with some $a \in \mathbb{R}$ and $b > 0$. Prove that u remains Gaussian for all times $t > 0$. (10)

We know that the fundamental solution is a Gaussian for every $t>0$. So we can start a multiple of the fundamental solution at some earlier time $t_0<0$ to match the coefficient b and adjust the prefactor to match a. This solution solves the heat equation, remains a Gaussian, and is the unique such solution in the class of solutions with inverse Gaussian bounds.

(Concrete implementation: since

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

choose t_0 such that $\frac{1}{4t_0} = b$ and $\frac{1}{(4\pi t_0)^{n/2}} A = a$

$$\Rightarrow A = \left(\frac{\pi}{b}\right)^{\frac{n}{2}} a .$$

Then $\phi(x,t) = A\,\Phi(x, t+t_0)$ satisfies $\phi(x,0) = a e^{-b|x|^2}$, solves the heat equation, and is a Gaussian for every $t \geq 0$.)

5. Recall that the solution to the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)\,,$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy\,,$$

where, for $t>0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|z|^2}{4t}}\,.$$

Assume that g is continuous and compactly supported. Show that there exists a $C>0$ such that

$$|Du(x,t)| \le \frac{C}{\sqrt{t}}\, \|g\|_{L^\infty}\,.$$

(10)

$$Du(x,t) = \int_{\mathbb{R}^n} D\Phi(x-y,t)\, g(y)\, dy = \int_{\mathbb{R}^n} D\Phi(y,t)\, g(x-y)\, dy$$

$$\Rightarrow |Du(x,t)| \leqslant \int_{\mathbb{R}^n} |\underbrace{D\Phi(y,t)}_{= -\frac{y}{2t(4\pi t)^{\frac{n}{2}}} e^{-\frac{|y|^2}{4t}}}|\, dy\, \|g\|_{L^\infty}$$

$$= \frac{1}{2t(4\pi t)^{n/2}} \underbrace{\int_{\mathbb{R}^n} |y|\, e^{\frac{-|y|^2}{4t}}\, dy}_{= \int_0^\infty r\, e^{-\frac{r^2}{4t}} \underbrace{\int_{\partial B(0,r)} dS}_{= n\alpha(n) r^{n-1}} dr} \|g\|_{L^\infty}$$

$$s = \frac{r}{\sqrt{t}} \Rightarrow ds = \frac{1}{\sqrt{t}}\, dr$$

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$$= n\alpha(n)\, t^{(n+1)/2} \int_0^\infty s^n e^{-\frac{s^2}{4}}\, ds$$

$$= \frac{C_1}{\sqrt{t}}\, \|g\|_{L^\infty}$$ as the final integral clearly converges.

6. Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $b \in C^1(\overline{U})$ satisfy

$$\operatorname{div} b \equiv D \cdot b = 0 \qquad \text{in } U,$$
$$\nu \cdot b = 0 \qquad \text{on } \partial U.$$

Further, suppose that $u \in C^1(\overline{U} \times [0,T])$ solves the transport equation

$$u_t + b \cdot Du = 0 \qquad \text{in } U.$$

(a) Show that

$$M = \int_U u \, dx$$

is constant in time.

(b) In a modeling scenario, u could describe the concentration of a certain substance in the container U. Give a corresponding physical interpretation of the result from (a). Further, what is the physical meaning of each of two conditions on b?

(10+10)

(a) $$\frac{dM}{dt} = \int_U u_t \, dx = -\int_U b \cdot Du \, dx = -\int_{\partial U} \underbrace{\nu \cdot b}_{=0} \, u \, dS + \int_U \underbrace{D \cdot b}_{=0} \, u \, dx$$

$$= 0 .$$

(b) If u denotes a concentration, then M denotes the total amount of the substance (e.g in units of mass).

$\frac{dM}{dt} = 0$ says that mass is preserved

b can be thought of the vector field describing the instantaneous velocity at each location in U. Then $\nu \cdot b = 0$ means that there is no mass or volume flux across the boundary. $\operatorname{div} b = 0$ means (by argument given in class) that volume elements preserve their volume as they are transported — no compression or expansion.

(Note: if volumes were not preserved, then transport of mass $\neq$ transport of concentration!)