1. Show that if v is a vertex of odd valency in a finite graph, then there exists a path connecting v to another vertex u of odd valency. (10)

Let C denote the connected component of the graph which contains v.

- As C is finite, the number of vertices in C with odd valency is even (proved in class), so C must contain at least one more vertex, u say, of odd valency.
- As C is connected, there exists a path connecting u and v.

2. Show that a tree with more than one vertex is bipartite. (10)

Fix any vertex v. As the graph is a tree, there is exactly one chain connecting v to any other vertex. (If there was another chain, the concatenation of the two would introduce a cycle!)

Now define

$$\ell(u) = \text{length of chain from } u \text{ to } v$$

and set

$$V_1 = \{ u \text{ vertex} : \ell(u) \text{ is even} \}$$

$$V_2 = \{ u \text{ vertex} : \ell(u) \text{ is odd} \}$$

which is clearly a disjoint partition of the vertex set. Moreover, when two vertices are connected by an edge, their ℓ-values must differ by one, i.e., there cannot be an edge between two vertices in V_1 or in V_2.

3. The structure of chemical molecules can be considered as a graph. Show that the graph of pentanol, an alcohol with the chemical formula $C_5H_{11}OH$, is a tree.

Note: Chemists know that Carbon (C), Oxygen (O), and Hydrogen (H) atoms have valencies 4, 2, and 1, respectively. Subscripts in the chemical formula denote the number of each atom in the molecule. (10)

$$|V| = 5 + 11 + 1 + 1 = 18$$

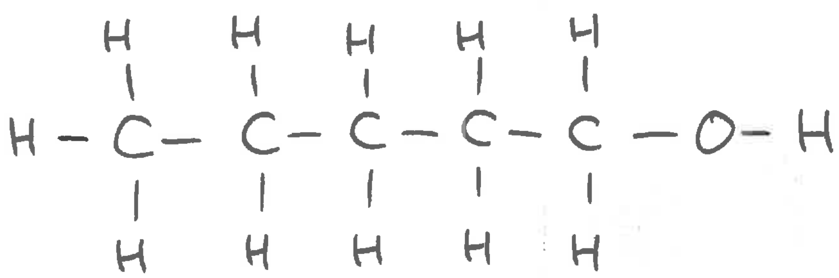
$$2|E| = \sum_{v \in V} \delta(v) = 5 \cdot 4 + 11 + 1 \cdot 2 + 1 = 34$$

$$\Rightarrow |E| = 17$$

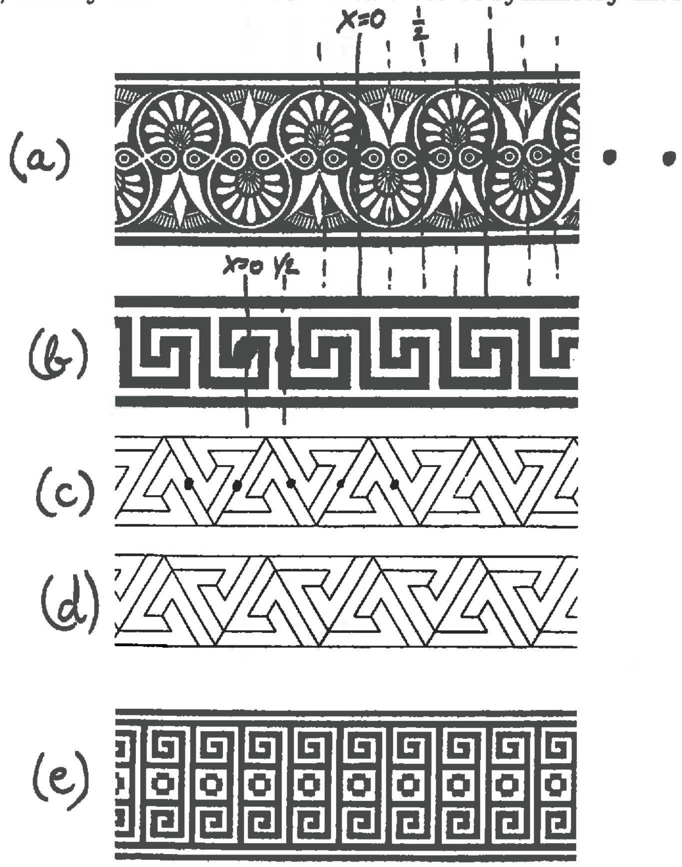
$$\Rightarrow \quad |F| = 2 - |V| + |E| = 1$$

$\Rightarrow$ It's a tree!

Remark: The typical configuration looks like



4. Identify the symmetries and corresponding frieze groups for the following five ornaments. Further, clearly draw all centers and lines of symmetry into the picture.



(From R.N. Umble, *Transformational Plane Geometry*, http://www.millersville.edu/~rumble/Math.355/Book/TPG-Spring2012.pdf.) (10)

(a) $G = \langle U, H_0 \rangle \cong D_\infty$. Note that it contains also $H_{\frac{n}{2}}$ and $R_{\frac{n}{2}+\frac{1}{4}}$, $n \in \mathbb{Z}$

(b) $G = \langle \tau, H_0 \rangle \cong D_\infty$. Note that G contains $H_{\frac{n}{2}}$, $n \in \mathbb{Z}$

(c) $G = \langle \tau, H_0 \rangle \cong D_\infty$ as in (b)

(d) $G = \langle U \rangle \cong \mathbb{Z}$

(e) $G = \langle \tau, R_{\parallel} \rangle \cong \mathbb{Z} \times \mathbb{Z}_2$

5. (a) Show that $V = \{1, 3, 5, 7\}$ endowed with multiplication modulo 8 is a group.

(b) Show that V is isomorphic to the dihedral group

$$D_2 = \langle a, b \mid a^2 = b^2 = e, ba = a^{-1}b \rangle .$$

(c) Can you think of a realization of this group as the symmetry group of some geometric shape?

4+3+3

(a) • V is closed under the group operation as the product of odd numbers is odd and any odd number modulo 8 is contained in V.

• Associativity follows from associativity of multiplication and the fact that multiplication and "modulo 8" commute.

• $e = 1$ (clear)

• $1 \cdot 1 = 1$, $3 \cdot 3 = 1$, $5 \cdot 5 = 1$, $7 \cdot 7 = 1$,

so each element is its own inverse.

(b) Set $a = 3$, $b = 5$. As $3 \cdot 5 = 7$, $ab = 7$

Since $(ab)^2 = e$, $aba = b \Rightarrow ba = ab = a^{-1}b$

Thus, all relations of D_2 are satisfied.

(c) It's the symmetry group of a non-square rectangle.

6. Let G be the group generated by Φ_A and Φ_B, two rotations about different centers of rotation A and B. Show that G contains a translation. (10)

Let R_{AB} denote the reflection about the line AB. Then we can find lines ℓ_1 and ℓ_2 s.t.

$$\Phi_A = R_{\ell_1} R_{AB} \quad \text{and} \quad \Phi_B = R_{AB} R_{\ell_2} .$$

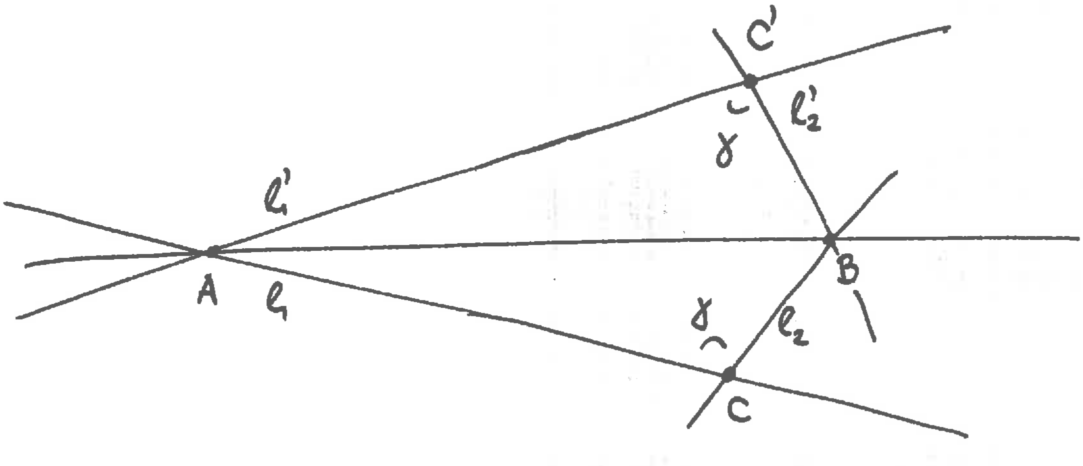
Let ℓ_1' denote the image of ℓ_1 under R_{AB}, likewise define ℓ_2'. Then

$$\Phi_A^{-1} = R_{\ell_1'} R_{AB} \quad \text{and} \quad \Phi_B^{-1} = R_{AB} R_{\ell_2'}$$

Then $\Phi_C := \Phi_A \Phi_B = R_{\ell_1} R_{\ell_2}$ is a rotation about the point C of intersection of ℓ_1 and ℓ_2,

likewise $\Phi_{C'} := \Phi_A^{-1} \Phi_B^{-1} = R_{\ell_1'} R_{\ell_2'}$ is a rotation about the point C' of intersection of ℓ_2' and ℓ_2'.

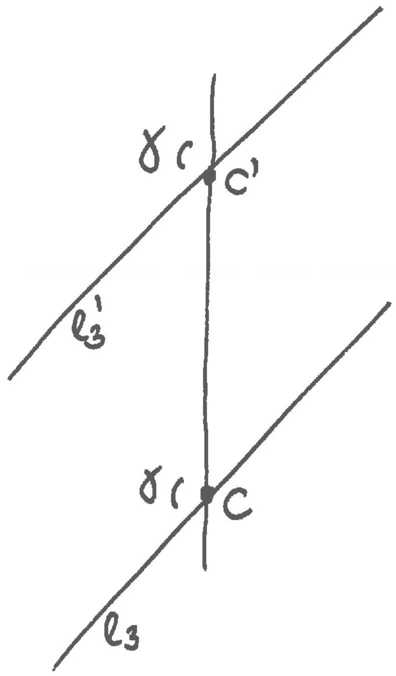
The angle of rotation for Φ_C is the negative of the angle for $\Phi_{C'}$.



Thus, we can represent these two rotations equivalently by

$$\Phi_C = R_{l_3} \; R_{CC'} \quad \text{and} \quad \Phi'_{C'} = R_{CC'} \; R_{l'_3}$$

where l_3 intersects $l_{CC'}$ at C and l'_3 is the translation of l_3 along $l_{CC'}$ to C'



We conclude that

$$\Phi_C \; \Phi'_{C'} = R_{l_3} R_{l'_3}$$

is a translation (l_3 and l'_3 are parallel).