Nonlinear Dynamics Lab

Session 7–9

due March 11, 2013

1 Introduction to numerical methods for ODEs

Note: The following are introductory exercises for getting acquainted with numerical methods for ordinary differential equations. They were partially covered in Session 7 and prepare for the Chua circuit project. No submission of results is required.

1. Consider the equation for the mathematical pendulum

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\sin q \,. \end{aligned}$$

   Write a program that solves the mathematical pendulum with

   (a) the explicit Euler method,

   (b) the implicit Euler method,

   (c) the implicit midpoint method.

   Take initial values $q(0) = 0$ and $p(0) = 1.9$ and plot example solutions $q(t)$ vs. t for each method into one graph for a time horizon of many oscillation periods.

2. Use your program from Exercise 1 to plot the solution trajectories in the q-p phase plane and overlay it with a contour plot of the energy

$$E = \frac{1}{2} p^2 - \cos q \,.$$

3. Read up on the build-in ODE solvers in `scipy.integrate.ode`. Obtain a reference solution for the mathematical pendulum using one of these integrators for a fixed final time, e.g. $T = 10$, and determine the order of the previously implemented methods by plotting the error (the difference of the result of your solver vs. that of the built-in reference solver) vs. the step size on a doubly logarithmic scale.

4. Solve the van der Pol equation

$$\dot{x} = y\,,$$
$$\dot{y} = \mu\,(1 - x^2)\,y - x$$

for $\mu = 1000$ with different built-in solvers. Be sure to involve at least one explicit solver and one implicit one.

2 Chua's Circuit

This weeks lab sessions study *Chua's Circuit*, a nonlinear electric oscillator which can be assembled with standard electronics parts. For background and circuit diagrams, consult the article by Hobson and Lansbury [1].

One of the crucial ingredients for the circuit is the inductor L. It is often necessary or convenient to use commodity inductors, which are typically far from ideal. In particular, real-world inductors have a resistance R_L which is typically non-negligible for the operation of the Chua oscillator. Other non-ideal effects include parasitic capacitance, radiative losses and hysteresis which we shall ignore to keep the modeling simple; however, these effects may still be significant.

To include the effect of the inductor's resistance into the mathematical description, we analyze it as a resistor in series with an ideal inductor. Consequently, the differential equations describing Chua's Circuit read

$$C_1\,\frac{\mathrm{d}V_1}{\mathrm{d}t} = \frac{V_2 - V_1}{R_\mathrm{c}} - I_\mathrm{nl}(V_1)\,, \tag{1}$$

$$C_2\,\frac{\mathrm{d}V_2}{\mathrm{d}t} = \frac{V_1 - V_2}{R_\mathrm{c}} + I_L\,, \tag{2}$$

$$L\,\frac{\mathrm{d}I_L}{\mathrm{d}t} = -V_2 - R_L\,I_L\,. \tag{3}$$

The resistance R_c is the coupling resistor, labeled $1/G$ in Hobson and Lansbury's article. A short derivation of these equations should be contained in your lab report.

3 Preparatory tasks

Note: Everybody should have made significant progress with the following tasks for the Monday lab session. They will be part of the lab report, to be submitted in groups of two. Code-sharing within the group is permitted. However, it is expected that every participant is familiar enough with the task that he/she could have written the code on their own and is able to run, modify, and explain the code on request.

1. Define a function for the nonlinear resistor whose transfer characteristic $I_\mathrm{nl}(V)$ is depicted in Figure 2 of Hobson and Lansbury. (Define parameters for the slopes and the

location of the kink; the computation of these values from the circuit diagram will be discussed on Monday.)

2. Code up a solver for the Chua system, either by modifying your solver from last week, or by using one of the build-in solvers from `scipy.integrate.ode`. (The latter is likely more robust and faster.) For values of the various parameters, see below

3. Something to think about: what should you plot in order to replicate the picture seen on the X-Y oscilloscope attached to the circuit as indicated? (This requires a bit of understanding of electrical circuits and will be discussed in detail on Monday. Don't worry if you cannot answer the question right now.)

4 Lab tasks

1. The inductor we have readily available is one with 10 mH. Take a multimeter and measure its resistance R_L.

2. Assemble Chua's circuit. Since the inductor is different from the one used in *Hobson and Lansbury*, some resistances should be chosen differently from the circuit diagram: take $R_1 = 2.2\,\mathrm{k}\Omega$ and $R_4 = 1\,\mathrm{k}\Omega$, the other values as in the diagram. For the variable resistance R_c we use a multiturn potentiometer which allows fine control of its resistance.

3. Determine experimentally the value of R_c which corresponds to the onset of chaos. Do you see a sharp transition, or a period doubling cascade as for the logistic map?

4. Measure the response curve of each of the nonlinear inverse resistors. To do so, use a third operational amplifier as a voltage follower attached to the noninverting inputs of the first two operational amplifiers. To the output of this voltage follower, you can attach the oscilloscope ground. The X- and Y-channels of the oscilloscope can then be attached to ground to measure $-V_1$ and to the far end of R_3 (or R_6). The voltage across R_3 resp. R_6 can be converted into the current response of the corresponding nonlinear resistor via Ohm's law.

5. If you have time: Use an inductor with 100 mH, also available in the lab, and try if you can—potentially modifying the values of the capacitors or resistors as well—find a regime as well. If this is successful, you will likely obtain less hysteresis in the response curve of the nonlinear resistors, and consequently better agreement of theory and experiment. (Note: the 100 mH inductor might have too big a resistance to excite nonlinear oscillations, so this is not guaranteed to work!)

5 Report items

1. Analyze the "inverse resistor" consisting of R_1, $R_2 = R_3$, and the operational amplifier. Write out an expression for the current response as a function of the input voltage, assuming that the operational amplifier saturates at output voltage $\pm V_{\text{max}}$. Write out and plot an expression for the overall response curve of the two parallel inverse resistors used in Chua's circuit.

2. Use the two experimentally response curves to obtain an approximate measured response curve. Plot the theoretical curve and the measured curve in one coordinate system.

3. Write a program which simulates Chua's circuit over a time interval $T = 0.02\,\text{s}$. Plot V_1 vs. V_2 for times $t = [T/2, T]$, thereby discarding transients. Find the value for R_c at the onset of chaos. (Both for the theoretical response curve and for the reconstruction of the measured curve where you may ignore hysteresis effects.)

4. Compare the experimental with the numerical results and discuss possible differences.

The experiment and the lab report may be done in groups of two.

6 Parts list

- 1 Capacitor 10 nF
- 3 Capacitors 100 nF
- 1 Inductor 10 mH
- 2 Resistors 220 Ω (rd-rd-bl-bl-br)
- 1 Resistor 1 kΩ (br-bl-bl-br-br)
- 1 Resistor 2.2 kΩ (rd-rd-bl-br-br)
- 2 Resistors 22 kΩ (rd-rd-bl-rd-br)
- 3 Operational Amplifiers LM741
- 1 Multi-turn potentiometer 1 kΩ

References

[1] P.R. Hobson and A.N. Lansbury, *A simple electronic circuit to demonstrate bifurcation and chaos*, Phys. Educ. **31** (1996), 39–43.