Partial Differential Equations

Final Exam

May 23, 2014, 12:30–14:30

1. Show that $u \in W^{1,\infty}(\mathbb{R}^n)$ if and only if $u\colon \mathbb{R}^n \to \mathbb{R}$ is bounded and Lipshitz continuous. (10+10)

2. Let $U \subset \mathbb{R}^n$ be open and bounded with C^2 boundary. Consider the bi-harmonic equation

$$\begin{aligned} \Delta^2 u = f \qquad & \text{in } U\,, \\ u = 0 \qquad & \text{on } \partial U\,, \\ \nu \cdot Du = 0 \qquad & \text{on } \partial U\,. \end{aligned}$$

   (a) Define a notion of weak solution for the bi-harmonic equation. Your answer should clearly state and define all the function spaces involved.

   (b) Prove the existence of a unique weak solution according to your definition.
   Hint: It may be useful to refer to the regularity theorem for uniformly elliptic second order problems, which says that a weak solution $v \in H_0^1(U)$ to

$$Lu = g \qquad \text{in } U$$

   satisfies

$$\|u\|_{H^2(U)} \leq c \left(\|u\|_{L^2(U)} + \|g\|_{L^2(U)} \right)$$

   for some constant c independent of g.

   (10+10)

3. Let

$$Lu = -\sum_{i,j=1}^{n} D_i(a^{ij}(x)\, D_j u)$$

   be a uniformly elliptic symmetric second order operator with bounded coefficients. Suppose $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies

$$Lu = 0$$

   in the sense of weak derivatives. Show that $u \in L^2(\mathbb{R}^n)$ implies that u is a constant. (10)

4. Let $U \in \mathbb{R}^n$ be open and bounded with smooth boundary, and $T > 0$. Prove that there is at most one smooth solution of the initial-boundary value problem

$$\begin{aligned} u_t - \Delta u = u^2 \qquad & \text{in } U_T\,, \\ \nu \cdot Du = 0 \qquad & \text{on } \partial U \times [0,T]\,, \\ u = g \qquad & \text{on } U \times \{t = 0\}\,. \end{aligned}$$

(10)

5. Let $U \in \mathbb{R}^n$ be open and bounded with smooth boundary. Suppose $u \in W^{1,\infty}(U, \mathbb{R}^n)$ with $\nu \cdot u = 0$ on ∂U and assume that $\theta = \theta(x,t)$ is a smooth solution to

$$\partial_t \theta + u \cdot D\theta = 0\,. \tag{*}$$

(a) Show that there exists a constant c such that

$$\|\theta(t)\|_{L^p}^p \le \mathrm{e}^{ct}\, \|\theta(0)\|_{L^p}^p$$

for every $2 \le p < \infty$ and $0 \le t < \infty$.

(b) Conclude that

$$\|\theta(t)\|_{L^\infty} \le \|\theta(0)\|_{L^\infty}\,.$$

Hint: You may use that

$$\|\theta\|_{L^\infty} = \lim_{p \to \infty} \|\theta\|_{L^p}\,.$$

(c) For every fixed $a \in U$ consider the autonomous ordinary differential equation

$$\frac{\mathrm{d}\phi(a,t)}{\mathrm{d}t} = u(\phi(a,t))\,.$$

Show that $\theta(x,t)$, implicitly defined via

$$\theta(\phi(a,t),t) = \theta(a,0)$$

solves (*). You may assume sufficient smoothness of all objects involved, and that $\phi(a,t) \in U$.

(d) Do you see a connection between the above and the result from Question 1? Explain.

(5+5+5+5)