Partial Differential Equations

Midterm Exam

March 27, 2014

1. Let $U \subset \mathbb{R}^n$ be open and bounded. Show that $H_0^1(U)$ is a strict subspace of $H^1(U)$. (5)

2. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Let $1 < p < \infty$ and define the Hölder conjugate p' by
$$\frac{1}{p} + \frac{1}{p'} = 1\,.$$
Show that $u \in W_0^{1,p}(U)$ if and only if $u \in L^p(U)$ and there exists a constant $C = C(u)$ such that
$$\left| \int_U u\, D\phi \,\mathrm{d}x \right| \leq C\, \|\phi\|_{L^{p'}(U)}$$
for all $\phi \in C^\infty(\mathbb{R}^n)$. ("only if" 5 + "if" 10)

   Hints: Note the class of test functions is arbitrary smooth functions on $\mathbb{R}^n$ restricted to U! Further, recall that $u \in W_0^{1,p}(U)$ if and only if $u \in W^{1,p}(U)$ and its trace is zero. (10)

3. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Let $h \in C^2(U) \cap C^1(\bar{U})$ and suppose further that there exists a constant $\theta > 0$ such that $h(x) \geq \theta$ and $\Delta h(x) \geq \theta$ for all $x \in U$.

   Show that the equation
$$\begin{aligned} -h\,\Delta u - 3\,Dh \cdot Du &= f \qquad \text{in } U\,, \\ u &= 0 \qquad \text{on } \partial U \end{aligned}$$
has a unique weak solution $u \in H_0^1(U)$ for every $f \in H^{-1}(U)$. (10)

4. State a condition on $f \in L^2(0,\pi)$ such that the equation
$$\begin{aligned} -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - u &= f(x) \qquad \text{on } (0,\pi)\,, \\ u(0) = u(\pi) &= 0 \end{aligned}$$
has a weak solution $u \in H_0^1(0,\pi)$. If it does, will the solution be unique? (10)

5. Let $c \in C(\mathbb{R}^n)$ with $c(x) \geq 1$ for all $x \in \mathbb{R}^n$. Let $u \in C^2(\mathbb{R}^n)$ satisfy

$$-\Delta u + c\, u = 1$$

and assume further that $\operatorname{supp}(Du)$ is compact.

Show that $u \leq 1$.

Hint: Consider the function $v = u - 1$. (10)