1. Let $U \subset \mathbb{R}^n$ be open and bounded. Show that $H_0^1(U)$ is a strict subspace of $H^1(U)$. (5)

The function $u = 1$ is contained in $H^1(U)$ but not in $H_0^1(U)$ as $Tu = 1$.

2. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Let $1 < p < \infty$ and define the Hölder conjugate p' by

$$\frac{1}{p} + \frac{1}{p'} = 1 .$$

Show that $u \in W_0^{1,p}(U)$ if and only if $u \in L^p(U)$ and there exists a constant $C = C(u)$ such that

$$\left| \int_U u \, D\phi \, dx \right| \leq C \, \|\phi\|_{L^{p'}(U)}$$

for all $\phi \in C^\infty(\mathbb{R}^n)$. ("only if" 5 + "if" 10)

Hints: Note the class of test functions is arbitrary smooth functions on $\mathbb{R}^n$ restricted to U! Further, recall that $u \in W_0^{1,p}(U)$ if and only if $u \in W^{1,p}(U)$ and its trace is zero. (10)

"$\Rightarrow$": Suppose first that $u \in C_0^\infty(U)$. Then

$$\left| \int_U u \, D\phi \, dx \right| = \left| \int_U Du \, \phi \, dx \right| \leq \underbrace{\|Du\|_{L^p}}_{\leq \|u\|_{W^{1,p}} = C} \|\phi\|_{L^{p'}} \quad \text{by Hölder}$$

Since $C_0^\infty(U)$ is dense in $W^{1,p}(U)$, this estimate extends to all $u \in W^{1,p}(U)$.

"$\Leftarrow$": As $L^{p'}$ is dual to L^p, we have

$$\|v\|_{L^p} = \sup_{\phi \in L^{p'}} \frac{\left| \int_U v\phi \, dx \right|}{\|\phi\|_{L^{p'}}} \underset{\uparrow}{=} \sup_{\phi \in C_0^\infty(U)} \frac{\left| \int_U v\phi \, dx \right|}{\|\phi\|_{L^{p'}}} \qquad (*)$$

C_0^∞ is dense in $L^{p'}$

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Now for fixed $u \in L^p(U)$ we can define $v \in L^p(U)$ by

$$\int_U u \, D\phi \, dx = - \int v \phi \, dx$$

Due to (*), we find that $\|v\|_{L^p} \leq C$ and, by construction, $v = Du$ in the weak sense.

$\Rightarrow u \in W^{1,p}(U)$

It remains to prove that $u \in W_0^{1,p}(U)$, i.e., that $Tu = 0$.

For any $\phi \in C^\infty(\mathbb{R}^n)$,

$$\int_U u\, D\phi\, dx = \int_{\partial U} Tu\, \phi\, \nu\, dS - \int_U Du\, \phi\, dx$$

(It's obvious for any mollified u, and we can pass to the limit due to the trace theorem.)

$$\Rightarrow \left| \int_{\partial U} Tu\, \phi\, \nu\, dS \right| \leq \left| \int_U u\, D\phi\, dx \right| + \left| \int_U Du\, \phi\, dx \right|$$

$$\leq C \|\phi\|_{L^{p'}} + \|Du\|_{L^p} \|\phi\|_{L^{p'}}$$

$$\leq \tilde{C}\, |\operatorname{supp} \phi|\, \|\phi\|_{L^\infty}$$

Now for every $\varepsilon > 0$ and any $g \in C(\partial U)$ it is possible to find $\phi \in C^\infty(U)$ s.t. $\|\phi - g\|_{L^\infty(\partial U)} < \frac{\varepsilon}{2}$ and $\tilde{C}\, |\operatorname{supp}\phi|\, \|\phi\|_{L^\infty} < \frac{\varepsilon}{2}$ (by choosing the support small enough).

$\Rightarrow Tu = 0$ by L^p duality on ∂U $\Rightarrow u \in W_0^{1,p}(U)$.

3. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Let $h \in C^2(U) \cap C^1(\bar{U})$ and suppose further that there exists a constant $\theta > 0$ such that $h(x) \geq \theta$ and $\Delta h(x) \geq \theta$ for all $x \in U$.

Show that the equation

$$-h\,\Delta u - 3\,Dh \cdot Du = f \qquad \text{in } U,$$
$$u = 0 \qquad \text{on } \partial U$$

has a unique weak solution $u \in H^1_0(U)$ for every $f \in H^{-1}(U)$. (10)

Associated bilinear form: Since, for $u, v \in C_0^\infty(U)$,

$$\int_U v(-h\Delta u - 3Dh \cdot Du)\,dx = \int_U (h\, Du \cdot Dv - 2\, v\, Dh \cdot Du)\,dx$$

we set, for $u, v \in H_0^1(U)$,

$$B(u,v) = \int_U (h\, Du \cdot Dv - 2\, v\, Dh \cdot Du)\,dx .$$

(i) $|B(u,v)| \leq \|h\|_{L^\infty} \|Du\|_{L^2} \|Dv\|_{L^2} + 2\, \|Dh\|_{L^\infty} \|v\|_{L^2} \|Du\|_{L^2}$

$\leq C(h)\, \|u\|_{H^1} \|v\|_{H^1}$ since $h \in C^2(U) \cap C^1(\bar{U})$.

$\Rightarrow$ B is continuous

(ii) $B(u,u) = \int_U (h\, |Du|^2 - 2\, u\, Dh \cdot Du)\,dx = \int_U (h\, |Du|^2 - Dh \cdot D|u|^2)\,dx$

$= \int_U (h\, |Du|^2 + \Delta h\, u^2)\,dx$

$\geq \theta\, \|u\|_{H^1}^2$ since $h \geq 0$, $\Delta h \geq 0$.

$\Rightarrow$ B is coercive.

$\Rightarrow$ By Lax-Milgram, the weak equation

$$B(u,v) = \langle f, v \rangle_{H^{-1}, H_0^1} \qquad \forall v \in H_0^1(U)$$

has a unique solution $u \in H_0^1(U)$ for every $f \in H^{-1}(U)$.

4. State a condition on $f \in L^2(0,\pi)$ such that the equation

$$-\frac{d^2u}{dx^2} - u = f(x) \qquad \text{on } (0,\pi),$$
$$u(0) = u(\pi) = 0$$

has a weak solution $u \in H_0^1(0,\pi)$. If it does, will the solution be unique? (10)

The operator $Lu = -u'' - u$ is clearly self-adjoint on $L^2(0,\pi)$.

Kernel of $L^* = L$ are solutions of the homogeneous problem

$$u'' = -u,$$
$$u(0) = u(\pi) = 0.$$

$u = \sin x$ is clearly one solution; it is the only one by standard ODE theory, or by using the Fourier series representation of u.

By Fredholm, $Lu = f$ has a solution if $f \perp \operatorname{Ker} L^*$, so if

$$\int_0^\pi f(x)\sin(x)\,dx = 0.$$

When the equation has a solution u, then any $v = u + \alpha \sin x$ is also a solution. ($\dim \operatorname{Ker} L^* = 1$)

5. Let $U \subset \mathbb{R}^n$ be open and bounded. Let $c \in C(\bar{U})$ with $c(x) \geq 1$ for all $x \in U$. Let $u \in C^2(U) \cap C(\bar{U})$ satisfy

$$-\Delta u + c u = 1 \quad , \qquad u = 0 \ \text{on } \partial U$$

Show that $u \leq 1$. (10)

Hint: Consider the function $v = u - 1$.

$$-\Delta v + cv + c = 1$$

$$\Rightarrow Lv \equiv -\Delta v + cv = 1 - c \leq 0$$

By the strong maximum principle, either

(i) $v = \text{const} \Rightarrow u = \text{const} \Rightarrow Lu = cu = 1$

$\Rightarrow u = \frac{1}{c} \leq 1$

or:

(ii) The interior max of v is negative, if it exists,

$\Rightarrow v \leq 0$

$\Rightarrow u \leq 1$