General Mathematics and Computational Science II

Midterm Exam

March 13, 2015

1. A bipartite graph with vertex sets V_1 and V_2 is *complete* if there is an edge between every vertex in V_1 and every vertex in V_2.

   Notation: The complete bipartite graph with $|V_1| = n$ and $|V_2| = m$ is called $K_{n,m}$. $K_{1,m}$, for any $m \in \mathbb{N}$, is called a *star*.

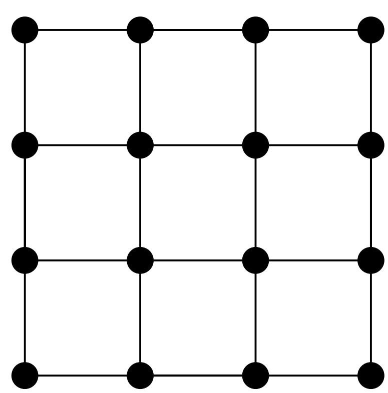
   (a) What is the number of edges of $K_{n,m}$?

   (b) Show that if a graph is complete bipartite and is a tree, then it is a star.

   (c) Show that $K_{3,3}$ cannot be embedded in the plane.

   (5+5+5)

2. In the following graph, find the shortest closed path that traverses every edge at least once.



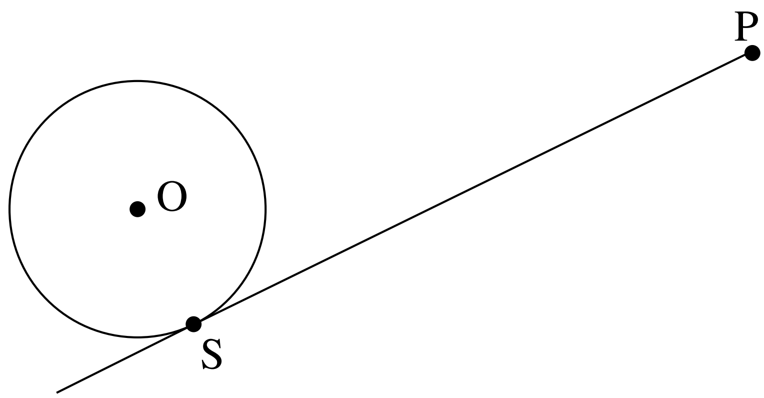
   Your answer should include

   (a) a correct solution;

   (b) an argument that there is no shorter such path.

   (5+5)

3. Given a circle with center O and a point P outside of it, suggest a construction using only Euclidean transformations (translation, reflection, rotation) to find a point S on the circle where the line segment PS is tangent to the circle. (10)



4. Given two lines ℓ_1 and ℓ_2, set $\ell_2' = R_{\ell_1}(\ell_2)$, i.e., ℓ_2' is the reflection of ℓ_2 about ℓ_1. Show that

$$R_{\ell_2}\, R_{\ell_1} = R_{\ell_1}\, R_{\ell_2'}\,.$$

(5)

5. For each of the following, determine whether or not it is a group. If it is a group, give a full argument why; if it is not, state at at least one property that fails.

   (a) The set of real $n \times n$ matrices under matrix multiplication;

   (b) the orientation-preserving motions of the plane under composition;

   (c) the orientation-reversing motions of the plane under composition.

   (A motion or Euclidean transformations of the plane is called orientation-preserving if a clock-wise traversal of a cycle is mapped onto clock-wise traversal of its image; it is orientation-reversing otherwise.) (4+3+3)

6. Let R_α denote the reflection about the line $x = \alpha$. Let G be the (symmetry) group generated by the unit translation along the x-axis and by R_0. Show that $R_\alpha \in G$ if and only if $2\alpha \in \mathbb{Z}$. (10)