1. A bipartite graph with vertex sets V_1 and V_2 is *regular* if every vertex in V_1 has degree d_1 and every vertex in V_2 has degree d_2.

   Show that there exists a matching from V_1 into V_2 provided $d_1 \geq d_2 > 0$. (10)

We will verify the condition for Hall's Theorem, e.g. in the form that $\forall\, S \subset V_1$,

$$|N(S)| \geq |S|$$

where $N(S) = \{v_2 \in V_2 : \exists \text{ edge between } v_2 \text{ and some } v_1 \in S\}$ denotes the graph neighborhood of S.

So let $S \subset V_1$ be arbitrary.

By definition, $N(S)$ has $d_1|S|$ incident edges connecting to S, and $d_2\,|N(S)|$ incident edges altogether.

$$\Rightarrow \quad d_2\,|N(S)| \geq d_1\,|S|$$

$$\Rightarrow \quad |N(S)| \geq \frac{d_1}{d_2}|S| \geq |S|$$

$\square$

2. Let H_A denote the point reflection about a point A with coordinates a. Let Π_v denote the translation by a vector v.

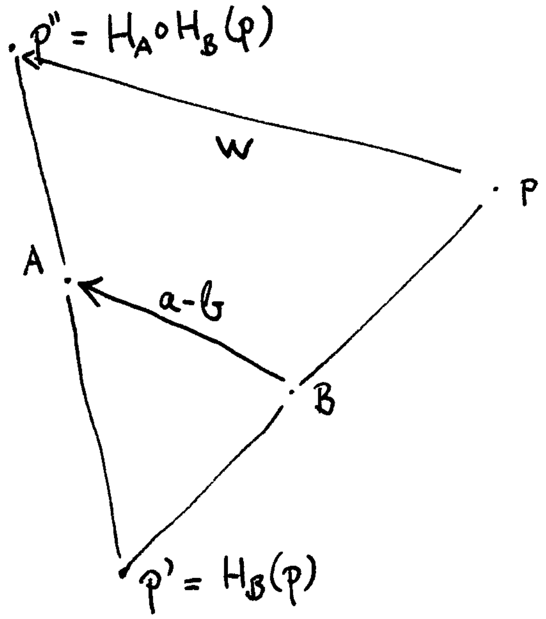
(a) Show that $H_A \circ H_B = \Pi_{2(a-b)}$.

(b) Show that $H_A \circ H_B = H_B \circ H_C$ if and only if B is the midpoint of the line segment AC.

(c) What can you say about the relative locations of A, B, C, and D in the case $H_A \circ H_B \circ H_C \circ H_D = \mathrm{Id}$?

(5+5+5)

(a)



$w = 2(a-b)$

since $\Delta ABp'$ and $\Delta p''pp'$ are similar and $|\overline{pp'}| = 2|\overline{Bp'}|$.

(b) $H_A \circ H_B = \Pi_{2(a-b)}$, $\quad H_B \circ H_C = \Pi_{2(b-c)}$

so equality of the two expressions is equivalent to

$$a-b = b-c$$

$$\Leftrightarrow b = \frac{a+c}{2}$$

(c) As above, the statement is equivalent to $a-b = d-c$, so the points are vertices of a parallellogram in clock-wise or anti-clockwise alphabetical order.

3. Let G_1 and G_2 be two finite Abelian groups, written additively. We define the direct sum $H = G_1 \oplus G_2$ as the set of ordered tuples $h = (g_1, g_2)$ with $g_1 \in G_1$ and $g_2 \in G_2$.

   (a) Define a +-operation on H that makes it an Abelian group.

   (b) Verify explicitly that with your definition you indeed obtain an Abelian group.

   (c) What is the order of H?

   (d) Let $\chi_i \colon G_i \to \mathbb{C}\setminus\{0\}$ be a character for $i = 1, 2$. Show that

$$\psi(h) = \chi_1(g_1) \cdot \chi_2(g_2)$$

   defines a character on H.
   (Recall that $\chi_i \colon G_i \to \mathbb{C}\setminus\{0\}$ is a character if it is a group homomorphism into $\mathbb{C}\setminus\{0\}$, the group of nonzero complex numbers endowed with multiplication.)

   (e) Show that every character on H is of this form.

(5+5+5+5+5)

(a) Define $(g_1, g_2) + (l_1, l_2) = (g_1 + l_1,\ g_2 + l_2)$

(b) Clearly, + is commutative, associative, and maps into H.

$0_H = (0_{G_1}, 0_{G_2})$ is the zero (neutral) element in H, as

$$(0_{G_1}, 0_{G_2}) + (g_1 + g_2) = (0_{G_1} + g_1,\ 0_{G_2} + g_2) = (g_1, g_2)$$

$(-g_1, -g_2)$ is the inverse of (g_1, g_2) with an equally obvious verification.

(c) $|H| = |G_1|\,|G_2|$

(d) $\psi(h + \tilde{h}) = \chi_1(g_1 + \tilde{g}_1) \cdot \chi_2(g_2 + \tilde{g}_2) = \chi_1(g_1) \cdot \chi_2(g_2) \cdot \chi_1(\tilde{g}_1) \cdot \chi_2(\tilde{g}_2)$

$= \psi(h) \cdot \psi(\tilde{h})$, so ψ is group homomorphism

(e) Let ψ be a character. Then $\psi(h) = \psi((g_1, 0) + (0, g_2)) = \psi((g_1, 0)) \cdot \psi((0, g_2))$

Now set $\chi_1(g_1) = \psi((g_1, 0))$ and $\chi_2 = \psi((0, g_2))$ which are characters on G_1, G_2.

4. Consider the linear programming problem

$$\text{maximize } z = 2x_1 + x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &\le 1, \\ x_1 - 2x_2 &\le 2, \\ x_1, x_2 &\ge 0. \end{aligned}$$

(a) Determine the solvability of this problem and find the solution if it exists, using the simplex method.

(b) Draw the feasible region and the level lines of the objective function; check for consistency with your answer in part (a).

(c) Write out the dual problem and determine its solvability. (No computation required.)

(10+5+5)

(a) Add slack variables to get initial tableaux:

x_1	x_2	s_1	s_2	
-1	1	1	0	1
1	-2	0	1	2
-2	-1	0	0	0

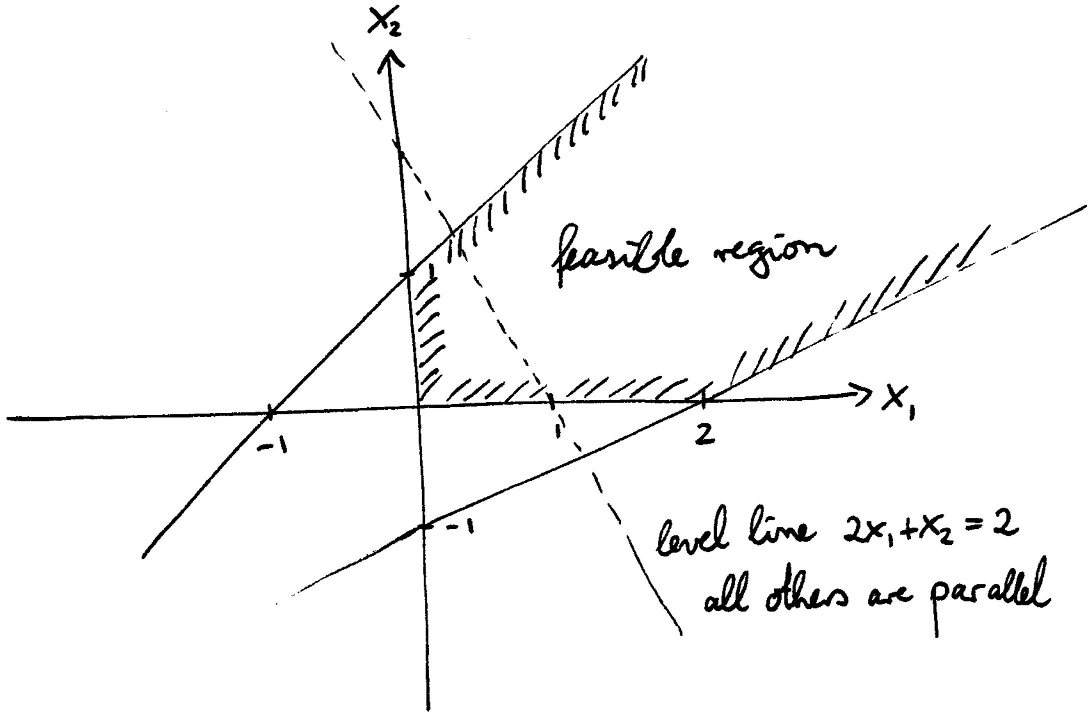
→

x_1	x_2	s_1	s_2	
0	-1	1	1	3
1	-2	0	1	2
0	-5	0	2	4

All elements in the x_2 - column are negative

⇒ feasible region is unbounded.

(b)



$\Rightarrow$ there is no finite maximum of z.

(c) Primal problem:

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

with $A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}$, $c^T = (-2, -1, 0, 0)$

$x = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Dual problem: maximize $y^T b$

subject to $A^T y \leq c$

By weak duality, its feasible region is empty.

5. Show that the problems

$$Ax = b, \qquad x \geq 0$$

and

$$y^{\mathsf{T}}A \geq 0, \qquad y^{\mathsf{T}}b < 0$$

cannot both have a solution. (10)

(In fact, *Farkas' lemma* states that excatly one of the two has a solution, but the complete proof is substantially more difficult.)

$$0 \leq y^T A x \qquad (\text{since } x \geq 0)$$

$$= y^T b \qquad (\text{by } Ax = b)$$

This contradicts $y^T b < 0$.

6. Recall that

$$\tilde{v}_k = \frac{1}{N}\sum_{j=0}^{N-1} e^{-ikjh}\, v_j$$

with $h = 2\pi/N$ denotes the discrete Fourier transform of the complex numbers $v_0, \dots, v_{N-1}$.

Define the discrete convolution of vectors $v = (v_0, \dots, v_{N-1})$ and $w = (w_0, \dots, w_{N-1})$ by

$$(v \circledast w)_j = \frac{1}{N}\sum_{\ell=0}^{N-1} v_\ell\, w_{j-\ell}$$

where all indices are understood modulo N.

Show that

$$\widetilde{(v \circledast w)}_k = \tilde{v}_k\, \tilde{w}_k\,.$$

(10)

$$\widetilde{(v \circledast w)}_k = \frac{1}{N}\sum_{j=0}^{N-1} \underbrace{e^{-ikjh}} \frac{1}{N}\sum_{\ell=0}^{N-1} v_\ell\, w_{j-\ell}$$

$$= e^{-ik(\ell+(j-\ell))h} = e^{-ik\ell h}\, e^{-ik(j-\ell)h}$$

$$= \frac{1}{N}\sum_{\ell=0}^{N-1} e^{-ik\ell h}\, v_\ell\, \underbrace{\frac{1}{N}\sum_{j=0}^{N-1} e^{-ik(j-\ell)h}\, w_{j-\ell}}_{=\frac{1}{N}\sum_{m=0}^{N-1} e^{-ikmh}\, w_m}$$

by periodicity!

$$= \tilde{v}_k\, \tilde{w}_k\,.$$

7. A 2π-periodic signal is sampled N times in the interval 2π. Suppose the signal contains only a fundamental frequency with wavenumber $k_0 = \frac{2}{5}N$ and its first harmonic at $k_1 = \frac{4}{5}N$.

   (a) Which frequencies does the trigonometric interpolant of the samples contain?

   (b) *Extra credit.* What does the reconstruction sound like? And what happens for different fundamental frequencies k_0?

(10+*10*)

Hint: Recall the aliasing formula

$$\tilde{u}_k = \sum_{m \in \mathbb{Z}} \hat{u}_{k-mN}$$

for $k = -N/2, \dots, N/2-1$.

(a) We have to find all $k \in \{-\frac{N}{2}, \dots, \frac{N}{2}-1\}$ s.t. $\exists\, m \in \mathbb{Z}$ with $k + mN \in \{k_0, k_1\}$.

For k_0, the frequency is already in the range of frequencies representable by the DFT. (Assume that N is a multiple of 5 for simplicity.) So $m=0$, there is no frequency shift. For k_1, we need to take $m=-1$, so it aliases to $k_1' = \frac{4}{5}N - N = -\frac{1}{5}N$

(b) The frequency ratio of the reconstruction is $\frac{\frac{2}{5}N}{\frac{1}{5}N} = 2$, so again an octave. Not too bad, it's adding a subharmonic. But for general nearby k_0 and its first harmonic $k_1 = 2k_0$, the aliased ratio is $\frac{k_0}{|k_1'|} = \frac{k_0}{|2k_0 - N|}$ which is almost arbitrary and can sound VERY bad.