1. Let $U \subset \mathbb{R}^n$ be open and bounded, and suppose that $u \in C^2(U) \cap C^1(\overline{U})$ solves

$$\begin{aligned} \Delta u = \lambda u \qquad & \text{in } U, \\ u = 0 \qquad & \text{on } \partial U \end{aligned}$$

for some $\lambda \geq 0$. Show that $u = 0$.

(In other words, show that the the eigenvalues of the Dirichlet Laplacian are strictly negative.)

Hint: Energy methods. (10)

Multiply by u and integrate:

$$\underbrace{\int_{\overline{U}} u \, \Delta u \, dx}_{} = \lambda \int_{\overline{U}} u^2 \, dx$$

$$= - \int_{\overline{U}} |Du|^2 \, dx + \underbrace{\int_{\partial U} u \, \nu \cdot Du \, dS}_{=0}$$

Hence, the LHS ≤ 0 while RHS ≥ 0, so they must be both 0.

$\Rightarrow$ $Du = 0$ pointwise

$\Rightarrow$ $u =$ const on each connected component of U

$\Rightarrow$ $u = 0$ as u vanishes on the boundary.

and simply connected

2. Let $U \subset \mathbb{R}^n$ be open and bounded, and suppose that $u \in C^2(U) \cap C^1(\bar{U})$ is harmonic. Show that if u has a zero in U, then u has a zero on ∂U. (10)

Due to the maximum principle, u takes its minimum and maximum value on $\partial \bar{U}$.

If $\partial \bar{U}$ is continuous and connected, then the intermediate value theorem implies that u has a zero on $\partial \bar{U}$.

<u>Note</u>: Connectedness of $\partial \bar{U}$ is crucial.

3. Let $U = (0,1) \times \mathbb{R}^{n-1}$.

(a) Show that for every $u \in C^1_c(\overline{U})$ with $u = 0$ on $\{x_1 = 0\} \times \mathbb{R}^{n-1}$,

$$\|u\|_{L^2(U)} \leq \sqrt{2}\,\|Du\|_{L^2(U)}\,.$$

Hint: Apply the fundamental theorem of calculus to $v = u^2$.

(b) Suppose b is a smooth divergence-free vector field on U, i.e., $D \cdot b = 0$. Let u be a smooth solution to the advection-diffusion equation

$$\begin{aligned} u_t + b \cdot Du &= \Delta u && \text{in } U \times (0,\infty)\,,\\ u &= 0 && \text{on } \partial U \times [0,\infty)\,,\\ u &= g && \text{on } U \times \{t=0\}\,. \end{aligned}$$

Use the result from part (a) to show that the energy $\|u\|^2_{L^2(U)}$ decays exponentially in time.

(c) Comment on the decay rate you observe in part (b) in relation to (i) estimates on the decay rate of the heat equation without advection and (ii) estimates on the decay rate for the heat equation on other domains you have encountered during this class.

(10+10+5)

(a) $$u^2(x) - \underbrace{u^2(0,x_2,\dots x_n)}_{=0} = \int_0^{x_1} \underbrace{\partial_{x_1} u^2(\xi, x_2, \dots, x_n)}_{= 2u(\xi,x_2,\dots,x_n)\, u_{x_1}(\xi,x_2,\dots x_n)}\, d\xi$$

$$\Rightarrow\quad u^2(x) \leq 2\int_0^{x_1} |u(\xi,x_2,\dots x_n)|\,|u_{x_1}(\xi,x_2,\dots x_n)|\,d\xi$$

$$\leq 2\int_0^1 |u(x)|\,|u_{x_1}|\,dx_1$$

Now integrate over $\bar{U}$:

Cauchy-Schwarz

$$\int_U u^2(x)\,dx \leq 2\int_U |u|\,|u_{x_1}|\,dx \underbrace{\int_0^1 dx_1}_{=1} \leq 2\,\|u\|_{L^2}\,\|u_{x_1}\|_{L^2}$$

$$\Rightarrow\quad \|u\|_{L^2} \leq 2\,\|u_{x_1}\|_{L^2} \leq 2\,\|Du\|_{L^2}$$

(b) Multiply by u and integrate:

$$\underbrace{\int_U u u_t \, dx}_{=\frac{1}{2}\frac{d}{dt}\int_U u^2 \, dx} + \underbrace{\int_U u \, b\cdot Du \, dx}_{=\frac{1}{2}\int_U b\cdot Du^2 \, dx} = \underbrace{\int_U u \Delta u \, dx}_{=-\int_U |Du|^2 \, dx} \quad \text{as } u=0 \text{ on } \partial U$$

$$= -\frac{1}{2}\int_U D\cdot b \, u^2 \, dx \quad \text{as } u=0 \text{ on } \partial U$$

$$= 0$$

Using the inequality from (a) on the RHS:

$$\frac{1}{2}\frac{d}{dt} \|u\|_{L^2}^2 \leq -\frac{1}{4} \|u\|_{L^2}^2$$

$$\Rightarrow \quad \|u(t)\|_{L^2}^2 \leq e^{-\frac{t}{2}} \|u(0)\|_{L^2}^2$$

(c) (i) We see that the advection term drops out completely, so volume preserving transport has no impact on the decay rate of total energy.

(ii) On $\mathbb{R}^n$, we only get algebraic decay via direct estimates on the fundamental solution (see Midterm exam, for example). We also looked at special bounded domains where we got exponential decay as here. This suggests that what matters is not whether the domain is bounded, but to have uniform bounds on the distance of every point to the boundary. (Which is true here, although U is unbounded!)

4. Consider Burgers' equation

$$u_t + u u_x = 0 \qquad \text{in } \mathbb{R} \times (0, \infty),$$
$$u = g \qquad \text{on } \mathbb{R} \times \{t = 0\},$$

where g is smooth and compactly supported. Find an expression for the first time of shock formation via the following sequence of steps.

(a) Write out a partial differential equation for $v \equiv u_x$.

(b) Set

$$\xi(t) = \underset{x \in \mathbb{R}}{\operatorname{argmin}}\, v(x, t) \qquad (*)$$

and show that $w(t) \equiv v(\xi(t), t)$ satisfies an ordinary differential equation.

(c) Solve this ordinary differential equation and deduce the time of shock formation from its solution.

(5+5+10)

(a) $U_{tx} + U_x^2 + U U_{xx} = 0$

$\Rightarrow V_t + V^2 + U V_x = 0$

(b) So long as the solution is smooth, V must have a critical point at the location of its minimum, i.e. $V_x(\xi(t), t) = 0$

$$\Rightarrow \dot{w} = \underbrace{V_x(\xi(t), t)}_{=0}\, \dot{\xi}(t) + \underbrace{V_t(\xi(t), t)}_{= -V^2(\xi(t),t) - U(\xi(t),t)\underbrace{V_x(\xi(t),t)}_{=0}}$$

$$= -w^2$$

(c) $\dfrac{dw}{w^2} = -dt \quad \Rightarrow \quad \displaystyle\int_{w(0)}^{w(t)} \frac{dw}{w^2} = -\int_0^t dt$

$$\Rightarrow + \frac{1}{w}\Big|_{w(0)}^{w(t)} = t \quad \Rightarrow \quad w(t) = \frac{1}{w(0)^{-1} + t}$$

So a shock forms no later than the denominator is zero,

so
$$t_{shock} \leqslant - \frac{1}{\min\limits_{x \in \mathbb{R}} g'(x)}$$

Notice that the argument is based on the assumption that $\xi(t)$ is differentiable. However, looking at the computation above, the result coincides with solving along the characteristic line $\dot{\xi}(t) = u(\xi(t), t)$ (**)

so that v_x along this characteristic is most rapidly decreasing amongst v_x along all other characteristics.

Thus, the definition of $\xi(t)$ via (*) and (**) coincide, and we conclude that

$$t_{shock} = - \frac{1}{\min\limits_{x \in \mathbb{R}} g'(x)} .$$

5. Recall that the Hopf–Lax solution to a Hamilton–Jacobi equation with Lagrangian L and initial condition g is given by

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}.$$

Show that when g is Lipshitz with constant λ i.e., if

$$|g(x) - g(y)| \leq \lambda |x-y| \qquad (*)$$

for all $x, y \in \mathbb{R}^n$, then u is Lipshitz with respect to x with the same constant L. (10)

$$u(x,t) - u(x',t) \leq tL\left(\frac{x-y}{t}\right) + g(y) - tL\left(\frac{x'-z}{t}\right) - g(z)$$

for every $y \in \mathbb{R}^n$, where z is the minimizer of $tL\left(\frac{x'-z}{t}\right) + g(z)$.

Now choose y such that $x - y = x' - z \Rightarrow y = x - x' + z$

$$\Rightarrow u(x,t) - u(x',t) \leq g(x - x' + z) - g(z)$$
$$\leq \lambda |x - x'| \qquad \text{by } (*)$$

Repeating the argument with x and x' exchanged then proves $|u(x,t) - u(x',t)| \leq \lambda |x - x'|$.

6. In a simple model for traffic flow along a one-way one-lane road, drivers will choose a velocity v that depends only on the traffic density ρ via

$$v(\rho) = 1 - \rho$$

(So the speed limit is normalized to $v = 1$ and the maximal density is $\rho = 1$ which corresponds to bumper-to-bumper traffic at stand-still.)

Then the flux of cars is given by

$$F(\rho) = \rho\, v(\rho)$$

and the car density follows the scalar conservation law

$$\rho_t + F(\rho)_x = 0\,.$$

(a) Explain that this conservation law expresses that the number of cars in any segment of road can change only via cars driving in at one end and leaving at the other.

(b) If the car density at time $t = 0$ is given by $\rho = g$, show that, in the absence of shocks, the traffic density is given by an expression of the form

$$\rho(x,t) = g(x - ct)$$

and write out an expression for the wave speed c.

(c) Show that the wave speed c can never exceed the driving velocity v. State a physical reason why this result must be true.

(d) Find the entropy solution to the "green light problem" where

$$g(x) = \begin{cases} 1 & \text{for } x \le 0 \\ 0 & \text{for } x > 0\,. \end{cases}$$

(e) Find the entropy solution to the "traffic jam problem" where traffic at maximal capacity hits upon a jammed up road. Specifically, take

$$g(x) = \begin{cases} \frac{1}{2} & \text{for } x \le 0 \\ 1 & \text{for } x > 0\,. \end{cases}$$

Hint: If you don't remember the Rankine–Hugoniot condition, it is easy to derive it in this case by common sense. (5+5+5+5+5)

(a) Let $N(t) = \int_a^b \rho(x,t)\,dx$ be the number of cars on the road segment $[a,b]$. Thus, by the FTC,

$$\dot{N} = F(\rho(a,t)) - F(\rho(b,t)),$$

so the rate of change of N is equal to the number of cars entering the segment at a per unit time minus the number of cars leaving the segment at b per unit time. No cars are entering or exiting in between.

(b) Solve along characteristic curves: if

$$z(s) = u(x(s), t(s))$$

then

$$z'(s) = u_x(x,t)\,\dot{x} + u_t(x,t)\,\dot{t} = 0$$

provided

$$\dot{t} = 1 \qquad \Rightarrow t = s$$
$$\dot{x} = F'(\rho(x,t)) \qquad \Rightarrow x = x_0 + F'(\rho(x,t))t$$
$$= x_0 + F'(g(x_0))\,t$$

and

$$u(x,t) = g(x_0) = g(x - ct)$$

with

$$c = F'(g(x_0))$$

(c) Look at c as a function of ρ:

$$c(\rho) = F'(\rho) = v(\rho) - \rho \leq v(\rho).$$

c is the speed of propagation of information in the system. As here all information is passed by moving cars, c can never exceed v.

(d) This is a rarefaction wave so the solution must be continuous for all $t > 0$. The characteristic line starting at $x = 0^-$ is propagating with speed $c_\ell = F'(1) = -1$, while the line at $x = 0^+$ is propagating with speed $c_r = F'(0) = 1$. We thus guess

$$\rho(x,t) = \begin{cases} 1 & \text{if } x + t \leq 0 \\ \frac{1}{2} - \frac{1}{2}\frac{x}{t} & \text{if } -t < x \leq t \\ 0 & \text{if } x - t > 0 \end{cases}$$

This expression can be verified by direct substitution. (Alternatively, use the formula for the solution of the Riemann problem.)

(e) The speed of the compression shock satisfies

$$[\![F(\rho)]\!] = \dot{\sigma} \, [\![\rho]\!],$$

$$\text{so } \dot{\sigma} = \frac{(1 - 1^2) - \left(\frac{1}{2} - \frac{1^2}{2}\right)}{1 - \frac{1}{2}} = -\frac{1}{2}$$

$$\Rightarrow \rho(x,t) = \begin{cases} \frac{1}{2} & \text{for } x \leq -\frac{1}{2}t \\ 1 & \text{for } x > -\frac{1}{2}t \end{cases}.$$

Note: If you cannot remember the Rankine-Hugoniot condition: The rate at which cars drive into the end of the jam is, seen from the side of the moving traffic, $\rho_\ell (v_\ell - \dot{\sigma})$. Seen from the side of the jam, it is $\rho_r(-\dot{\sigma})$. As these must be equal, $\frac{1}{2}\left(\frac{1}{2} - \dot{\sigma}\right) = -\dot{\sigma} \Rightarrow \dot{\sigma} = -\frac{1}{2}$.