Introductory Partial Differential Equations

Midterm Exam

March 23, 2015

1. Solve the inhomogeneous transport equation

$$u_t + u_x + t\,\mathrm{e}^{-x} = 0 \qquad \text{in } \mathbb{R} \times (0, \infty)\,,$$
$$u(x,0) = \mathrm{e}^{-x} \qquad \text{on } \mathbb{R}\,.$$

(10)

2. Suppose $u \colon \mathbb{R}^n \to \mathbb{R}$ is harmonic with

$$\int_{\mathbb{R}^n} |u| \, dx < \infty\,.$$

Show that this implies $u = 0$. (5)

3. Suppose u is a harmonic function on $\mathbb{R}^n$ with $n \geq 2$ such that

$$u(x) = x_1 + x_2 \qquad \text{on } \partial B(0,1)\,.$$

(a) What are the minimum and maximum values of u on $\overline{B(0,1)}$?

(b) Find $u(0)$.

(5+5)

4. Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $u \in C^2(\overline{U})$ be a solution to the *Helmholtz equation* with Neumann boundary conditions,

$$-\Delta u + u = f \qquad \text{in } U\,,$$
$$\nu \cdot Du = g \qquad \text{on } \partial U\,.$$

(a) Show that u is the unique such solution.

(b) What can you say about uniqueness of solutions to the Poisson equation $\Delta u = f$ in otherwise the same setting?

(5+5)

5. Recall that the solution to the heat equation

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \mathbb{R} \times (0, \infty)\,, \\ u &= g \quad \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

is given by

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\, g(y)\, dy\,,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{-\frac{|z|^2}{4t}}\,.$$

(a) Show that for every $g \in L^1(\mathbb{R})$ there exists a constant $c > 0$ such that

$$\sup_{x \in \mathbb{R}} |u(x,t)| \leq \frac{c}{\sqrt{t}}\,.$$

(b) Show that for every $g = h_x$ with $h \in L^1(\mathbb{R})$ there exists a constant $c > 0$ such that

$$\sup_{x \in \mathbb{R}} |u(x,t)| \leq \frac{c}{t}\,.$$

(c) Give a qualitative explanation for (b) vs. (a).

(5+5+5)

6. Consider the equation[1]

$$\begin{aligned} u_t - u_{xxt} + 3\,u\,u_x &= 2\,u_x\,u_{xx} + u\,u_{xxx} \qquad \text{in } \mathbb{R} \times (0, \infty)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,. \end{aligned}$$

Suppose that u is a solution such that $u(x,t) \to 0$ as $x \to \pm\infty$ with u_x and u_{xx} bounded for every fixed $t \geq 0$.

Show that

$$M(t) = \int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx$$

is a constant of the motion. (10)

[1] This equation is most commonly known as the Camassa–Holm equation, though its derivation goes back to Fokas and Fuchssteiner.