Nonlinear Dynamics Lab

Session 2 and 3

due February 15, 2016

1. Consider the logistic map
$$x_{i+1} = r\, x_i\, (1 - x_i)$$
Write a program to plot the bifurcation diagram in the range $r \in [2, 4]$.

   Hints:

   - Use vectorized code to compute the iterates for *all* values of r simultaneously.
   - Divide the x-interval $[0, 1]$ into some number of bins and keep a count of the number of times each bin is visited by the trajectory.
   - To ensure that your trajectory is sufficiently close to the asymptotic state, you need to compute some number of pre-iterations—iterations of the logistic map which are not included into the bin count. The initial value will not matter. For example, $x_0 = \frac{1}{2}$ will do just fine.
   - Use `imshow` to display the histogram so created as an image.

2. Estimate the capacity dimension of the logistic map
$$x_{n+1} = r\, x_n\, (1 - x_n)$$
at the critical value of the bifurcation parameter $r = 3.569945672$.

   Hints:

   - The *capacity dimension* (also known as *fractal dimension*) is defined as
$$d_{\text{capacity}} = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \varepsilon^{-1}}\,,$$
where ε is the size of each bin from the previous exercise and $N(\varepsilon)$ is the total number of bins which have a nonzero bin count for a given bin size ε.

     However, this limit is not accessible computationally. Instead, create a doubly logarithmic plot (use `loglog` for plotting) so that the capacity dimension arises as the slope of the region of the graph in which you can see a clear linear dependence between $\log \varepsilon$ and $\log N(\varepsilon)$.

- The most efficient way to find $N(\varepsilon)$ for varying values of epsilon is to first compute the bin-counts as in Exercise 1 on the finest level with $N_{\text{bins}} = 2^k$ for some integer k, then obtain bin-counts at successively coarser scales by repeated pairwise adding of the bin-counts of neighboring bins.

3. After you have found computational parameters so that you can compute the capacity dimension at the critical parameter in a stable manner, extend your code so that it computes the capacity without user intervention. Plot the capacity dimension as a function of r for values of r around the critical value.

You should submit by the deadline:

- For Exercise 1, the program code electronically. The program code must be runnable and produce the requested output without further user intervention.
- For Exercise 2, the code as well as the doubly logarithmic plot together with a very brief discussion of how you extracted the capacity dimension from the plot.
- For Exercise 3, the program code which must be runnable and produce the requested output without further user intervention. Moreover, you must briefly explain the choice of computational parameters (maximal number of bins, smallest value of ε chosen, number of pre-iterations, number of iterations, and other potentially relevant choices). Discuss the sensitivity of your result on the choice of these computational parameters.