

# Operations Research

## Midterm Exam

April 1, 2016

1. Use the graphical method to minimize

$$z = 4x + 3y$$

subject to

$$2x + 3y \geq 6,$$

$$3x - 2y \leq 9,$$

$$x + 5y \leq 20,$$

$$x, y \geq 0.$$

(20)

2. The Steelworks have three machines each of which has  $t_i$  hours of spare operating time available. They can be used to make four products, each with possible profit of  $p_j$  Euros per unit sold. The production of one unit of product  $j$  requires  $w_{ij}$  hours of working time on machine  $i$ .

The production schedule for the week has been planned using the attached Pyomo program. Look at the program output to answer the following questions.

- (a) Marketing determines that product P02 will still sell if the unit price is raised by 20 Euros. Give an estimate of the resulting increase in profits.
- (b) On Tuesday, three hours before the end of the last shift, the cutting machine breaks down. The service engineer needs the rest of the day to fix the machine, but is certain that it will be back in service the next morning. Will this affect the planned weekly output?
- (c) One of the workers would like to borrow the welding machine to mend his fender. His boss is inclined to grant the request, but insists that the cost in lost profits to the company must be covered by the borrower. How much should he charge if the machine is used for two hours?
- (d) In an alternative scenario, the welding machine is out for 10 hours. Do you expect that the loss in profits is 5 times your answer to question (b)? What do you advise?

(5+5+5+5)

```
In [1]: from pyomo.environ import *
        from pyomo.opt import *
        opt = solvers.SolverFactory("glpk")
```

```
In [2]: t = {'Cut': 38, 'Roll': 28, 'Weld': 21}
        p = {'P01': 260, 'P02': 350, 'P03': 250, 'P04': 370}
        w = {('Cut', 'P01'): 1.1,
              ('Cut', 'P02'): 2.5,
              ('Cut', 'P03'): 1.7,
              ('Cut', 'P04'): 2.6,
              ('Roll', 'P01'): 1.7,
              ('Roll', 'P02'): 2.1,
              ('Roll', 'P03'): 1.4,
              ('Roll', 'P04'): 2.4,
              ('Weld', 'P01'): 1.6,
              ('Weld', 'P02'): 1.3,
              ('Weld', 'P03'): 1.6,
              ('Weld', 'P04'): 0.8}

        P = list(p.keys())
        M = list(t.keys())

        model = ConcreteModel()
        model.x = Var(P, within=NonNegativeReals)

        def profit_rule(model):
            return sum(p[j]*model.x[j] for j in P)
        model.profit = Objective(rule=profit_rule, sense=maximize)

        def capacity_rule(model, i):
            return sum(w[i,j]*model.x[j] for j in P) <= t[i]
        model.capacity_constraint = Constraint(M, rule=capacity_rule)

        model.dual = Suffix(direction=Suffix.IMPORT)
        results = opt.solve(model)
```

```
In [3]: model.x.get_values()
```

```
Out[3]: {'P01': 0.0, 'P02': 10.0, 'P03': 5.0, 'P04': 0.0}
```

```
In [4]: model.profit.expr()
```

```
Out[4]: 4750.0
```

```
In [5]: [(i,
          model.dual[model.capacity_constraint[i]],
          model.capacity_constraint[i].uslack())
         for i in M]
```

```
Out[5]: [('Cut', 0.0, 4.5),
          ('Roll', 152.597402597403, 0.0),
          ('Weld', 22.7272727272728, 0.0)]
```

3. Prove that if there are feasible solutions for both primal and dual form of a linear programming problem, then both must have an optimal solution. (10)
4. Andy, Bob, and Curt together with Doris, Erica, and Fiona visit a desert island. Each woman is spending a certain fraction of her time with one of the men and none alone; likewise, each man is spending a certain fraction of his time with one of the women and none alone. The following table gives a measure of the happiness that ensues from each possible couple spending time together.

	Doris	Erica	Fiona
Andy	1	3	3
Bob	4	3	2
Curt	5	4	2

- (a) Formulate a linear programming problem to maximize overall happiness on the island, assuming that happiness is additive and proportional to the fraction of time each couple spends together.  
(You do not need to solve this problem.)
- (b) Do you expect the result to suggest monogamous or polygamous relationships? Explain.

(10+5)

5. Recall that the primal form of the standard activity analysis problem

$$\begin{aligned} & \text{maximize } \mathbf{c}^T \mathbf{x} \\ & \text{subject to } A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned} \tag{P}$$

has the symmetric dual

$$\begin{aligned} & \text{minimize } \mathbf{y}^T \mathbf{b} \\ & \text{subject to } \mathbf{y}^T A \geq \mathbf{c}^T, \\ & \mathbf{y} \geq 0. \end{aligned} \tag{D}$$

Here,  $A$  is an  $n \times m$  matrix,  $\mathbf{x}, \mathbf{c} \in \mathbb{R}^m$ , and  $\mathbf{y}, \mathbf{b} \in \mathbb{R}^n$ .

Now a certain type of application leads to the linear programming problem

$$\begin{aligned} & \text{maximize } s \\ & \text{subject to } s \leq B\mathbf{p}, \\ & p_1 + \cdots + p_\ell = 1, \\ & \mathbf{p} \geq 0, \end{aligned} \tag{P'}$$

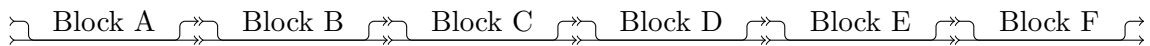
where  $B$  is a given  $k \times \ell$  matrix. There are  $\ell + 1$  decision variables, namely  $s \in \mathbb{R}$  and  $\mathbf{p} \in \mathbb{R}^\ell$ . Note that there is no sign constraint on  $s$  and there is one *equality* constraint.

- (a) Write this problem in the standard form (P) by introducing an artificial non-negative variable and representing the equality by two inequalities. In particular, write out  $A$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  for a given matrix  $B$  and state their dimensions.
- (b) Show that problem (P') also has a symmetric dual formulation, namely

$$\begin{aligned}
 & \text{minimize } t \\
 & \text{subject to } t \geq B^T \mathbf{q}, \\
 & \quad q_1 + \cdots + q_k = 1, \\
 & \quad \mathbf{q} \geq 0.
 \end{aligned}
 \tag{D'}$$

(5+10)

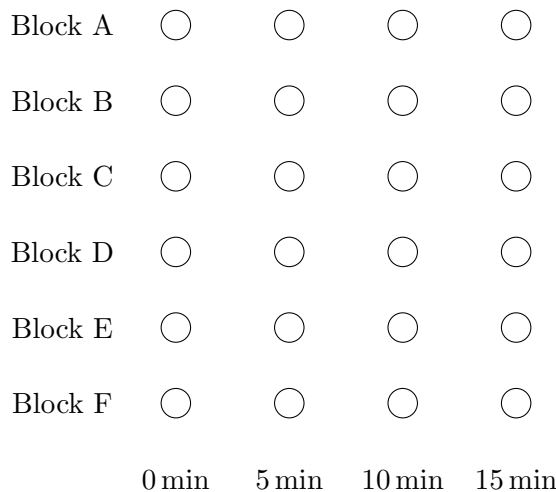
6. A railway line is used for passenger and freight traffic in a single direction. The line is composed of 6 signal blocks. Each block can be used by only one train at a time, but at the start of each signal block there is a siding where another train can be kept waiting.



For simplicity, we analyze the problem in discrete 5-minute time slots. Trains enter a signal block exactly at the start of each slot.

A freight train takes exactly 5 minutes to traverse a signal block. Passenger trains travel twice as fast and must not be delayed. Thus, the block entered by a passenger train *as well as one block ahead* must be free of moving freight trains, but there could be a freight train kept waiting in one of the sidings.

This problem can be analyzed as a maximum flow problem for the movement of freight trains on a network whose nodes represent the signal blocks at the start of each time slot.



At time  $t = 0$  min, a passenger train enters the the first block A. The schedule repeats every 15 minutes, so that the occupancy of blocks at  $t = 0$  min and  $t = 15$  min, as well as the respective in- and outflows must coincide.

- (a) Mark the signal blocks occupied by the passenger train in the network graph.
- (b) Indicate with arcs the possible movement of freight trains in the network. Each node in the top layer can receive unrestricted inflow, each node at the bottom layer can have an outflow subject to the restrictions explained above.
- (c) How many freight trains can enter the line during each 15 minute period?
- (d) How much time do to the freight trains spend waiting in a siding?

(5+5+5+5)