Introduction to Partial Differential Equations

Homework 3

due March 22, 2017

1. Evans, p. 87 problem 12.
2. Evans, p. 87 problem 13.
3. Let $u(x,t)$ solve the heat equation

$$\begin{aligned} u_t - \Delta u = 0 \qquad & \text{in } \mathbb{R}^n \times (0,\infty)\,, \\ u = g \qquad & \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

   with $g \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Show that

$$\|u\|_{L^\infty} \to 0 \qquad \text{as } t \to \infty$$

   while

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \text{const}\,.$$

   Give a physical interpretation of each of the statements.

4. Find a solution formula for the heat equation with advection,

$$\begin{aligned} u_t - \Delta u + b \cdot Du = 0 \qquad & \text{in } \mathbb{R}^n \times (0,\infty)\,, \\ u = g \qquad & \text{on } \mathbb{R}^n \times \{t=0\}\,. \end{aligned}$$

   Hint: which equation is solved by $v(x,t) = u(x+bt,t)$?

5. Evans, p. 88 Problem 17.
6. For $h \in C^2(\mathbb{R}^3)$ with compact support and define, for $t > 0$,

$$u(x,t) = \fint_{\partial B(x,t)} t\,h(y)\,dS(y)\,.$$

   Show that

   (a) There exists $C > 0$ such that $|u(x,t)| \le C/t$ for all $t > 0$.

(b) $\lim_{t \to 0} u(x,t) = 0$.

(c) $\lim_{t \to 0} u_t(x,t) = h(x)$.

(d) u solves the wave equation

$$u_{tt} - \Delta u = 0$$

on $\mathbb{R}^3 \times (0, \infty)$.