Homework 3 Solutions

1. (a) $\frac{dy}{dt} = -4\frac{t}{y}$

$$\Rightarrow \quad y\,dy = -4t\,dt$$

$$\Rightarrow \quad \int_{y_0}^{y(t)} y\,dy = -4\int_0^t t\,dt$$

$$\Rightarrow \quad \frac{1}{2}y^2\Big|_{y_0}^{y(t)} = -2t^2\Big|_0^t$$

$$\Rightarrow \quad y^2(t) - y_0^2 = -4t^2$$

$$\Rightarrow \quad y(t) = \pm\sqrt{y_0^2 - 4t^2}$$

<u>Note</u>: To ensure that $y(0) = y_0$, take the sign which corresponds to the sign of y_0!

The solution fails to exist (as a real-valued function) if $y_0^2 - 4t^2 < 0$, i.e., $|y_0| < 2t_0$

$\Rightarrow$ The interval of existence is $[0, \frac{1}{2}|y_0|]$.

(b) $\frac{dy}{dt} = -y^3$

$$\Rightarrow \quad \frac{dy}{y^3} = -dt \qquad \Rightarrow \quad \int_{y_0}^{y(t)} \frac{dy}{y^3} = -\int_0^t dt$$

$$\Rightarrow \quad -\frac{1}{2} y^{-2} \Big|_{y_0}^{y(t)} = -t$$

$$\Rightarrow \quad \frac{1}{y^2(t)} - \frac{1}{y_0^2} = 2t$$

$$\Rightarrow \quad y^2(t) = \frac{1}{2t + y_0^{-2}}$$

$$\Rightarrow \quad y(t) = \pm\sqrt{\frac{1}{2t + y_0^{-2}}}$$

As in (a), choose the sign that corresponds to the sign of y_0.

- If $y_0 \neq 0$, solution exists for all $t \geq 0$.
- If $y_0 = 0$, the (unique) solution is $y(t) = 0$, so in this case, the solution also exists for all $t \geq 0$.

2. (a) $\dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt}$ (chain rule)

$$= (1-n)\, y^{-n} \left(-py + q\, y^{n}\right)$$

$$= (1-n)\left(-p\, y^{1-n} + q\right)$$

$$= (n-1)\left(pv - q\right)$$

(b) $y' + \frac{2}{t} y = \frac{1}{t^2} y^3$

$\Rightarrow$ $v = y^{-2}$ satisfies the equation

$$\frac{dv}{dt} = 2\left(\frac{2}{t} v - \frac{1}{t^2}\right) \qquad v(1) = \frac{1}{y^2(1)} = 1$$

$$\Rightarrow \frac{dv}{dt} - \frac{4}{t} v = -\frac{2}{t^2}$$

Integrating factor: $e^{\int_1^t \frac{-4}{t} dt} = e^{-4 \ln t \big|_1^t} = t^{-4}$

$$\Rightarrow \frac{d}{dt}\left(v t^{-4}\right) = -\frac{2}{t^6}$$

$$\Rightarrow v(t) \frac{1}{t^4} - v(1) = -2 \int_1^t \frac{dt}{t^6} = -2 \frac{1}{-5} t^{-5} \Big|_1^t$$

$$= \frac{2}{5}\left(\frac{1}{t^5} - 1\right)$$

$$\Rightarrow v(t) = t^4 v(1) + t^4 \frac{2}{5}\left(\frac{1}{t^5} - 1\right)$$

$$= t^4 - \frac{2}{5} t^4 + \frac{2}{5} \frac{1}{t} = \frac{3}{5} t^4 + \frac{2}{5} \frac{1}{t}$$

$$\Rightarrow y(t) = \frac{1}{\sqrt{\frac{3}{5} t^4 + \frac{2}{5} \frac{1}{t}}}$$

(choose +ve root due to initial condition!)

(c) $y' - ry = -\frac{r}{K} y^2$

$\Rightarrow v = \frac{1}{y}$ solves

$$\frac{dv}{dt} = -rv + \frac{r}{K} \qquad V_0 = v(0) = \frac{1}{y_0}$$

$$\Rightarrow \frac{d}{dt}\left(e^{rt} v\right) = \frac{r}{K} e^{rt}$$

$$\Rightarrow e^{rt} v(t) - v(0) = \frac{1}{K}\left(e^{rt} - 1\right)$$

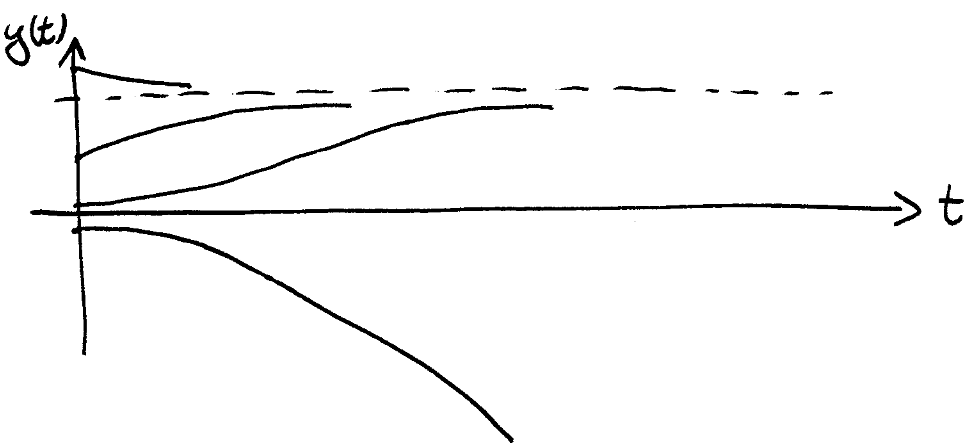
$$\Rightarrow v(t) = e^{-rt} v(0) + \frac{1}{K}\left(1 - e^{-rt}\right)$$

$$\Rightarrow y(t) = \frac{1}{\frac{1}{K} + e^{-rt}\left(\frac{1}{y_0} - \frac{1}{K}\right)}$$

3.(a) $y' = y - y^2 = 0 \Rightarrow y(1-y) = 0 \Rightarrow y = 0$ or $y = 1$

@ $y=0$, y' changes sign from $-$ to $+$ $\Rightarrow$ unstable

@ $y=1$, y' changes sign from $+$ to $-$ $\Rightarrow$ stable

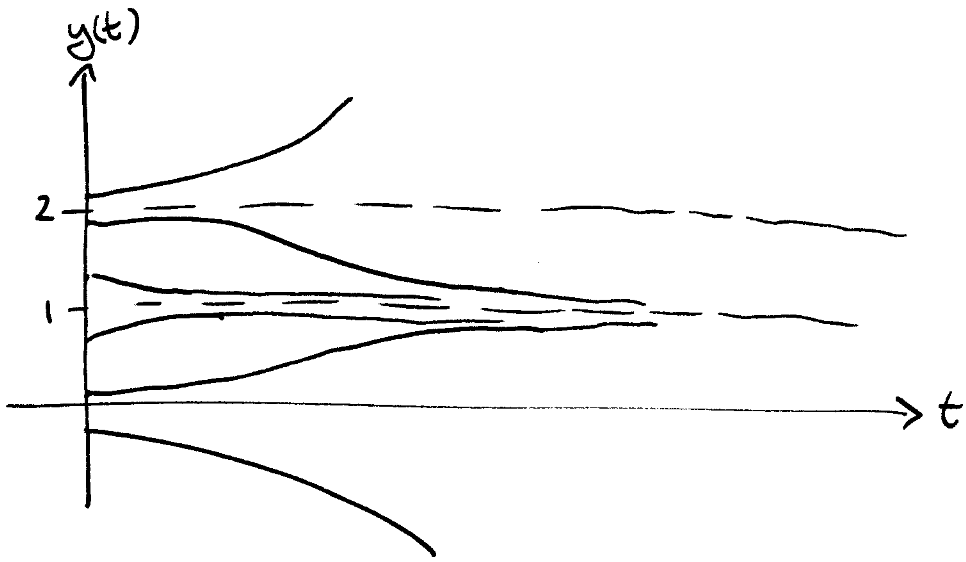


(b) $y' = y(y-1)(y-2) = 0 \Rightarrow y=0$ or $y=1$ or $y=2$

@ $y=0$: y' changes sign from $-$ to $+$ $\Rightarrow$ unstable

@ $y=1$: y' changes sign from $+$ to $-$ $\Rightarrow$ stable

@ $y=2$: y' changes sign from $-$ to $+$ $\Rightarrow$ unstable



(c) $y' = e^y - 1 = 0 \Rightarrow e^y = 1 \Rightarrow y = 0$

@ $y=0$, y' changes sign from $-$ to $+$ $\Rightarrow$ unstable