Applied Differential Equations and Modeling

Homework 3

Due in class Thursday, March 1, 2018

1. For each of the following initial value problems, find the solution and state how the interval of existence of the solution depends on the initial value y_0.

   (a) $y' = -4\,\dfrac{t}{y}\,, \qquad y(0) = y_0$

   (b) $y' + y^3 = 0\,, \qquad y(0) = y_0$

2. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form
$$y' + p(t)\,y = q(t)\,y^n$$
and is called Bernoulli's equation.

   (a) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation.

   (b) Use this method to solve the initial value problem
$$t^2\,y' + 2t\,y - y^3 = 0\,, \quad y(1) = 1\,.$$

   (c) Use this method to find, by a different method than the one explained in class, the solution to the logistic differential equation
$$y' = r\,y\left(1 - \frac{y}{K}\right), \quad y(0) = y_0\,.$$

3. For each of the following equations, determine the equilibrium points where $y'(x) = 0$ and classify each as stable (y' changes sign from positive to negative at x) or unstable (y' changes sign from negative to positive at x). Sketch a few solution curves (without trying to solve the equation) in the t-y plane.

   (a) $y' = y - y^2$

   (b) $y' = y\,(y-1)\,(y-2)$

   (c) $y' = \mathrm{e}^y - 1$

4. Let $P(t)$ denote the number of fish in a lake at time t, and let C denote the "carrying capacity" of the lake. Suppose further that fishermen catch a fraction k of the fish per unit of time, so that the population satisfies the equation

$$\frac{dP}{dt} = \left(1 - \frac{P}{C}\right) P - k\, P \qquad \text{with} \qquad P(0) = P_0\,.$$

(a) For given values of C and k, when is the population increasing, when is it decreasing?

(b) For given values of C and k, how many fish will be in the lake in the long run? (You do not need to solve the differential equation to answer this question!)

(c) Which value of the fishing rate k maximizes the number of fish caught in the long run?

(d) Solve the differential equation explicitly and check that your solution is consistent with your answer to part (b).