1. Consider the system

$$\dot{x} = \mu + x^2 + x\,y + y^2\,,$$
$$\dot{y} = x^2 + x\,y - y\,,$$
$$\dot{\mu} = 0\,.$$

(a) Consider the critical point $(0,0,0)$ and compute the linearly stable, linearly unstable, and linearly neutral subspaces.

(b) Note that the system has a two-dimensional center manifold. Compute an expression for the center manifold up to terms of second order.

(c) Show that the motion on the center manifold is given by

$$\dot{u} = \mu + u^2 + \text{h.o.t.}\,,$$
$$\dot{\mu} = 0\,.$$

(d) Now treat μ as a parameter and show that the system in center manifold coordinates has a bifurcation at $\mu = 0$. Name that bifurcation and sketch a bifurcation diagram in center manifold coordinates. (Extra credit if you manage to sketch the bifurcation in the original $2+1$-dimensional phase space.)

(5+5+5+5)

(a) At $(x,y,\mu) = (0,0,0)$, the Jacobian matrix is

$$DF = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that, by inspection, the system has a one-dimensional stable subspace $V_s = \text{span}\{e_2\}$ with eigenvalue -1 and a two-dimensional neutral gen. eigenspace $V_c = \text{span}\{e_1, e_3\}$.

(b) We can parameterize the center manifold W_c by x and μ.

Since it is tangent to the x-μ plane, the parameterization must start with quadratic terms, i.e.

$$y = a\,x^2 + b\,x\mu + c\,\mu^2 + \text{h.o.t.}$$

$$\Rightarrow \quad \dot{y} = 2ax\dot{x} + b\dot{x}\mu + bx\dot{\mu} + 2c\mu\dot{\mu}$$

$$= (2ax + b\mu)(\mu + x^2 + xy + y^2)$$

on the other hand:

$$\dot{y} = x^2 + (x-1)(ax^2 + bx\mu + c\mu^2)$$

$$\Rightarrow \quad 2ax\mu + b\mu^2 = x^2 - ax^2 - bx\mu - c\mu^2 + \text{h.o.t.}$$

$$\Rightarrow \quad a = 1, \quad b = -2, \quad c = 2$$

(c) Insert back into the x-equation (with $x \equiv u$):

$$\dot{u} = \mu + u^2 + \text{h.o.t.}$$

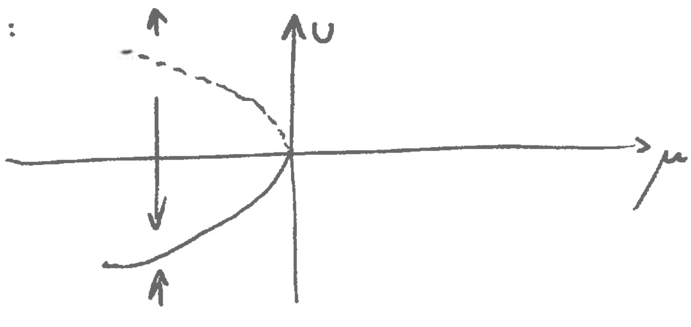
(The μ-equation remains unchanged)

(d) Critical points: $\mu + u^2 = 0$ (up to h.o.t.)

$$\Rightarrow \quad u = \pm\sqrt{-\mu}$$

I.e. no critical point for $\mu > 0$, $u = 0$ is crit. pt. for $\mu = 0$, when $\mu < 0$, $u = \sqrt{-\mu}$ is unstable, $u = -\sqrt{-\mu}$ is stable.

It is a saddle-node:



2. Consider the system

$$\dot{x} = \mu x + y - x\,(x^2 + y^2)\,,$$
$$\dot{y} = -x + \mu y - y\,(x^2 + y^2)\,.$$

follows most directly from (b) and (c).

(a) Show that the origin is the only critical point and determine the linear stability of the origin as a function of the real parameter μ.

(b) Write the system in polar coordinates.

(c) Show that the system has a periodic orbit for $\mu > 0$. Conclude that the system has an Andronov–Hopf-bifurcation at $\mu = 0$.

(d) Sketch the bifurcation diagram.

(5+5+5+5)

(a) At $(x,y) = (0,0)$, the Jacobian matrix is

$$DF = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$$

with eigenvalues given by $(\mu - \lambda)^2 + 1 = 0$

$\Rightarrow \mu - \lambda = \pm i \quad \Rightarrow \quad \lambda = \mu \pm i$

So for $\mu < 0$, the system is linearly stable, for $\mu = 0$, it is linearly neutrally stable, and for $\mu > 0$ it is linearly unstable.

(b) Write $x = r\cos\theta$, $y = -r\sin\theta$

$\Rightarrow \dot{r}\cos\theta + r\dot{\theta}(-\sin\theta) = \mu r\cos\theta - r\sin\theta - r^3\cos\theta$

$-\dot{r}\sin\theta - r\dot{\theta}\cos\theta = -r\cos\theta - \mu r\sin\theta + r^3\sin\theta$

$\Rightarrow \dot{r} = \mu r - r^3$

$-r\dot{\theta} = -r$, so $\dot{\theta} = 1$

(c) Critical points of the r- equation:

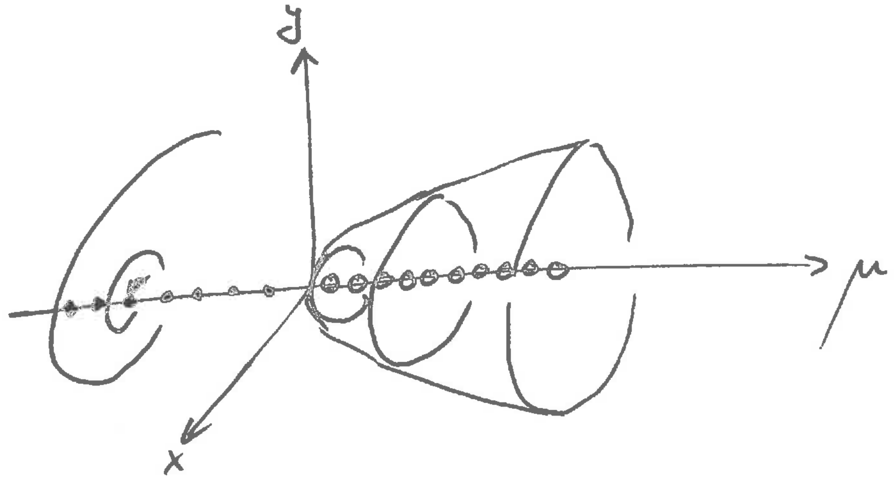
$$\mu r - r^3 = 0$$

$\Rightarrow r = 0$ or $\mu = r^2$

When $\mu > 0$, there is a pair of critical points at $r = \pm\sqrt{\mu}$, clearly these are stable, since $\dot{\Theta} = 1$, it's a per. orbit in x-y variables

The critical point at $r = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$ $\Rightarrow$ Andronov-Hopf bifurcation at $\mu = 0$.

(d)



For $\mu < 0$, there is a stable focus, for $\mu > 0$ an unstable focus and a stable periodic orbit.

3. Prove the following simple averaging theorem.

Consider the equation

$$\dot{x} = \varepsilon\, f(x,t) + \varepsilon^2\, g(x,t;\varepsilon)\,,$$

where $f\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is T-periodic for fixed $x \in \mathbb{R}^n$. Assume that f and g are smooth. Suppose that there exists a ball of radius $R > 0$ on which f, g, and all their derivatives are bounded independent of ε.

Define

$$\overline{f}(y) = \frac{1}{T}\int_0^T f(y,t)\,\mathrm{d}t$$

and suppose that the solution of

$$\dot{y} = \varepsilon\,\overline{f}(y) \qquad y(0) = x(0)$$

remains in a ball of radius $R/2$. Then

$$x(t) - y(t) = O(\varepsilon)$$

on the time scale $1/\varepsilon$.

Hint: Define

$$u(x,t) = \int_0^t (f(x,s) - \overline{f}(x))\,\mathrm{d}s\,,$$

set

$$x = z + \varepsilon\, u(z,t)$$

and derive a differential equation for z. (10)

Follow the hint:

$$\dot{x} = \dot{z} + \varepsilon \dot{u} = \dot{z} + \varepsilon\left(f(z,t) - \overline{f}(z)\right) + \varepsilon\, Du\, \dot{z}$$

on the other hand,

$$\dot{x} = \varepsilon\, f(z + \varepsilon u, t) + \varepsilon^2 g$$

Note that, by definition, u is bounded uniformly in ε and t.

The same argument applies to Du.

Thus, $x = z + O(\varepsilon)$ for as long as z remains in B_R.

Further, $g = O(\varepsilon^2)$ and $f(z+\varepsilon v, t) = f(z,t) + O(\varepsilon)$

for as long as x and z remain in B_R.

$$\Rightarrow \quad \left(I + \varepsilon D v\right)\dot{z} = \varepsilon \bar{f}(z) + O(\varepsilon^2)$$

Under the assumptions above, and for ε small enough, $I + \varepsilon D v$ is uniformly invertible with

$$\left(I + \varepsilon D v\right)^{-1} = I - \varepsilon D v + O(\varepsilon^2)$$

$$\Rightarrow \quad \dot{z} = \varepsilon \bar{f}(z) + O(\varepsilon^2)$$

so that, by the Gronwall lemma, $z = y + O(\varepsilon)$ on the time scale $\frac{1}{\varepsilon}$. We conclude that

$$x = y + O(\varepsilon)$$

on the time scale $\frac{1}{\varepsilon}$.

Note that the argument above is contingent on x and z remaining in B_R. However, since y remains in $B_{\frac{R}{2}}$ by assumption and x, z remain $O(\varepsilon)$-close, they cannot leave B_R on the time scale $\frac{1}{\varepsilon}$ if ε is small enough.

4. Consider *Mathieu's equation*

$$\ddot{x} + (1 + 2\varepsilon\, \cos(2t))\, x = 0\,.$$

(a) Use the ansatz

$$x = z_1\, \cos t + z_2\, \sin t\,, \qquad (*)$$
$$\dot{x} = -z_1\, \sin t + z_2\, \cos t \qquad (**)$$

to show that

$$\left.\begin{aligned} \dot{z}_1 &= 2\varepsilon\, \sin t\, \cos(2t)\, (z_1\, \cos t + z_2\, \sin t)\,, \\ \dot{z}_2 &= -2\varepsilon\, \cos t\, \cos(2t)\, (z_1\, \cos t + z_2\, \sin t)\,. \end{aligned}\right\} (***)$$

(b) Use averaging (see attached table for useful trigonometric identities!) to conclude that

$$x(t) = \tfrac{1}{2}\,(x(0) + \dot{x}(0))\, \mathrm{e}^{-\frac{1}{2}\varepsilon t}\,(\cos t + \sin t) + \tfrac{1}{2}\,(x(0) - \dot{x}(0))\, \mathrm{e}^{\frac{1}{2}\varepsilon t}\,(\cos t - \sin t) + O(\varepsilon)$$

on the time scale $1/\varepsilon$. (Check the assumptions of the statement in Problem 3!)

(10+10)

(a) Differentiate $(*)$ and use $(**)$:

$$\dot{x} = \dot{z}_1 \cos t - z_1 \sin t + \dot{z}_2 \sin t + z_2 \cos t$$
$$\equiv - z_1 \sin t + z_2 \cos t$$

Differentiate $(**)$ and use the equation:

$$\ddot{x} = -\dot{z}_1 \sin t - z_1 \cos t + \dot{z}_2 \cos t - z_2 \sin t$$
$$\equiv -(1 + 2\varepsilon \cos 2t)(z_1 \cos t + z_2 \sin t)$$

$$\Rightarrow \quad \dot{z}_1 \cos t + \dot{z}_2 \sin t = 0$$
$$-\dot{z}_1 \sin t + \dot{z}_2 \cos t = -2\varepsilon \cos 2t\,(z_1 \cos t + z_2 \sin t)$$

$$\Rightarrow \quad \dot{z}_1 = 2\varepsilon \sin t \cos 2t \,(z_1 \cos t + z_2 \sin t)$$

$$\dot{z}_2 = -2\varepsilon \cos t \cos 2t \,(z_1 \cos t + z_2 \sin t)$$

(b) $\sin t \cos t \cos 2t = \frac{1}{2} \sin 2t \cos 2t = \frac{1}{4} \sin 4t$

$\rightarrow$ averages to 0

$$\sin^2 t \cos 2t = \frac{1}{2}(1 - \cos 2t) \cos 2t$$

$$= \frac{1}{2} \cos 2t - \frac{1}{4}(1 + \cos 4t)$$

$\rightarrow$ averages to $-\frac{1}{4}$

$$\cos^2 t \cos 2t = \frac{1}{2}(1 + \cos 2t) \cos 2t$$

$$= \frac{1}{2} \cos 2t + \frac{1}{4}(1 + \cos 4t)$$

$\rightarrow$ averages to $\frac{1}{4}$

$\Rightarrow$ the averaged system reads

$$\dot{y}_1 = -\frac{1}{2}\varepsilon y_2$$

$$\dot{y}_2 = -\frac{1}{2}\varepsilon y_1 \qquad \text{or} \qquad \dot{y} = \begin{pmatrix} 0 & -\frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & 0 \end{pmatrix} y$$

This linear system has eigenvalues $\lambda_1 = -\frac{\varepsilon}{2}$ and $\lambda_2 = \frac{\varepsilon}{2}$

with eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$\Rightarrow \quad y(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{\varepsilon}{2}t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\frac{\varepsilon}{2}t}$$

$$\Rightarrow \quad x(t) = \left(c_1 e^{-\frac{\varepsilon}{2}t} + c_2 e^{\frac{\varepsilon}{2}t}\right) \cos t + \left(c_1 e^{-\frac{\varepsilon}{2}t} - c_2 e^{\frac{\varepsilon}{2}t}\right) \sin t$$

$$\dot{x}(t) = -\left(c_1 e^{-\frac{\varepsilon}{2}t} + c_2 e^{\frac{\varepsilon}{2}t}\right) \sin t + \left(c_1 e^{-\frac{\varepsilon}{2}t} - c_2 e^{\frac{\varepsilon}{2}t}\right) \cos t$$

At $t=0$:

$$\left.\begin{aligned} x(0) &= c_1 + c_2 \\ \dot{x}(0) &= c_1 - c_2 \end{aligned}\right\} \Rightarrow c_1 = \frac{x(0)+\dot{x}(0)}{2}, \quad c_2 = \frac{x(0)-\dot{x}(0)}{2}$$

Inserting these expressions into the solution for $x(t)$, we obtain the result as claimed.

It remains to verify that the assumptions of the averaging theorem are satisfied. We look at (***). Structure, periodicity, and smoothness match trivially. It remains to check boundedness of the averaged solution. Indeed, for $t = \frac{1}{\varepsilon}$,

$$\|y(t)\| \leq |c_1| \sqrt{2}\, e^{-\frac{1}{2}} + |c_2| \sqrt{2}\, e^{\frac{1}{2}}.$$

Choosing R twice this RHS, we can proceed.

5. Consider the system

$$\left.\begin{aligned} \varepsilon\,\dot{y} &= z + g(\phi)\,, \\ \varepsilon\,\dot{z} &= -y\,, \end{aligned}\right\} \Rightarrow \; -\varepsilon^2 \ddot{z} = z + g(\phi)$$

$$\dot{\phi} = I\,,$$

$$\dot{I} = 0\,,$$

where $\phi \in \mathbb{T}$, where $\mathbb{T} = [0, 2\pi]$ with identification of the endpoints.

Show that the y-z dynamics can be parameterized via $z = z(\phi, I)$ provided the resonance condition

$$I \neq \frac{1}{\varepsilon k}\,, \quad k \in \mathbb{Z}$$

is satisfied.

Hint: Fix I, expand $g(\phi)$ and $z(\phi, I)$ as a Fourier series, and match coefficients. (10)

The angle ϕ is 2π-periodic, so we can write

$$g(\phi) = \sum_{k \in \mathbb{Z}} g_k\, e^{ik\phi}$$

$$z(\phi, I) = \sum_{k \in \mathbb{Z}} z_k\, e^{ik\phi} \quad \Rightarrow \quad \ddot{z} = \sum_{k \in \mathbb{Z}} z_k \,(-k^2 I^2)\, e^{ik\phi}$$

$$\Rightarrow \quad \varepsilon^2 \sum_{k \in \mathbb{Z}} z_k\, k^2 I^2 e^{ik\phi} = \sum_{k \in \mathbb{Z}} z_k\, e^{ik\phi} + \sum_{k \in \mathbb{Z}} g_k\, e^{ik\phi}$$

$$\Rightarrow \quad \varepsilon^2 k^2 I^2 z_k = z_k + g_k \quad \text{for every } k \in \mathbb{Z}$$

$$\Rightarrow \quad g_k = \frac{z_k}{1 - \varepsilon^2 k^2 I^2}$$

So we can solve for g_k provided $1 - \varepsilon^2 k^2 I^2 \neq 0$,

i.e. if $I \neq \frac{1}{\varepsilon k}$ for any $k \in \mathbb{Z}$.