Ordinary Differential Equations

Final Exam

May 22, 2018

1. Consider the system

$$\dot{x} = \mu + x^2 + x\,y + y^2\,,$$
$$\dot{y} = x^2 + x\,y - y\,,$$
$$\dot{\mu} = 0\,.$$

   (a) Consider the critical point $(0, 0, 0)$ and compute the linearly stable, linearly unstable, and linearly neutral subspaces.

   (b) Note that the system has a two-dimensional center manifold. Compute an expression for the center manifold up to terms of second order.

   (c) Show that the motion on the center manifold is given by

$$\dot{u} = \mu + u^2 + \text{h.o.t.}\,,$$
$$\dot{\mu} = 0\,.$$

   (d) Now treat μ as a parameter and show that the system in center manifold coordinates has a bifurcation at $\mu = 0$. Name that bifurcation and sketch a bifurcation diagram in center manifold coordinates. (Extra credit if you manage to sketch the bifurcation in the original $2 + 1$-dimensional phase space.)

(5+5+5+5)

2. Consider the system

$$\dot{x} = \mu\,x + y - x\,(x^2 + y^2)\,,$$
$$\dot{y} = -x + \mu\,y - y\,(x^2 + y^2)\,.$$

   (a) Show that the origin is the only critical point and determine the linear stability of the origin as a function of the real parameter μ.

   (b) Write the system in polar coordinates.

   (c) Show that the system has a periodic orbit for $\mu > 0$. Conclude that the system has an Andronov–Hopf-bifurcation at $\mu = 0$.

(d) Sketch the bifurcation diagram.

(5+5+5+5)

3. Prove the following simple averaging theorem.

   Consider the equation
$$\dot{x} = \varepsilon\, f(x,t) + \varepsilon^2\, g(x,t;\varepsilon)\,,$$
   where $f\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is T-periodic for fixed $x \in \mathbb{R}^n$. Assume that f and g are smooth. Suppose that there exists a ball of radius $R > 0$ on which f, g, and all their derivatives are bounded independent of ε.

   Define
$$\overline{f}(y) = \frac{1}{T}\int_0^T f(y,t)\,\mathrm{d}t$$
   and suppose that the solution of
$$\dot{y} = \varepsilon\,\overline{f}(y) \qquad y(0) = x(0)$$
   remains in a ball of radius $R/2$. Then
$$x(t) - y(t) = O(\varepsilon)$$
   on the time scale $1/\varepsilon$.

   Hint: Define
$$u(x,t) = \int_0^t (f(x,s) - \overline{f}(x))\,\mathrm{d}s\,,$$
   set
$$x = z + \varepsilon\, u$$
   and derive a differential equation for z. (10)

4. Consider *Mathieu's equation*
$$\ddot{x} + (1 + 2\varepsilon\,\cos(2t))\,x = 0\,.$$

   (a) Use the ansatz
$$x = z_1\,\cos t + z_2\,\sin t\,,$$
$$\dot{x} = -z_1\,\sin t + z_2\,\cos t$$
   to show that
$$\dot{z}_1 = 2\varepsilon\,\sin t\,\cos(2t)\,(z_1\,\cos t + z_2\,\sin t)\,,$$
$$\dot{z}_2 = -2\varepsilon\,\cos t\,\cos(2t)\,(z_1\,\cos t + z_2\,\sin t)\,.$$

(b) Use averaging (see attached table for useful trigonometric identities!) to conclude that

$$x(t) = \tfrac{1}{2}\,(x(0) + \dot{x}(0))\,\mathrm{e}^{-\frac{1}{2}\varepsilon t}\,(\cos t + \sin t) + \tfrac{1}{2}\,(x(0) - \dot{x}(0))\,\mathrm{e}^{\frac{1}{2}\varepsilon t}\,(\cos t - \sin t) + O(\varepsilon)$$

on the time scale $1/\varepsilon$. (Check the assumptions of the statement in Problem 3!)

(10+10)

5. Consider the system

$$\begin{aligned} \varepsilon\,\dot{y} &= z + g(\phi)\,, \\ \varepsilon\,\dot{z} &= -y\,, \\ \dot{\phi} &= I\,, \\ \dot{I} &= 0\,, \end{aligned}$$

where $\phi \in \mathbb{T}$, where $\mathbb{T} = [0, 2\pi]$ with identification of the endpoints.

Show that the y-z dynamics can be parameterized via $z = (\phi, I)$ provided the resonance condition

$$I \neq \frac{1}{\varepsilon k}\,, \quad k \in \mathbb{Z}$$

is satisfied.

Hint: Fix I, expand $g(\phi)$ and $z(\phi, I)$ as a Fourier series, and match coefficients. (10)