Ordinary Differential Equations

Midterm Exam

April 5, 2018

1. Consider the system

$$\dot{x} = -x + y^2\,,$$
$$\dot{y} = -y^3 + x^2\,.$$

   (a) Find all critical points and determine their linear stability.

   (b) Clearly, the origin is a critical point where linear stability analysis is inconclusive. Show that the origin is an asymptotically stable equilibrium point.
   Hint: Show that $V(x,y) = x^2\,(1+y) + y^2$ is a Lyapunov function in some neighborhood of the origin.
   You may find the following elementary inequality useful:

$$a\,b \leq \frac{\delta}{2}\,a^2 + \frac{1}{2\delta}\,b^2$$

   for all $a, b \in \mathbb{R}$ and $\delta > 0$.

   (c) Sketch the phase portrait.

(10+10+5)

2. Consider the n-dimensional linear system

$$\dot{x} = A(t)x$$

where $A(t)$ is T-periodic. Recall that the Floquet theorem states that a fundamental matrix solution to this equation can be written as

$$\Phi(t) = P(t)\,\mathrm{e}^{Bt}$$

where $P(t)$ is a T-periodic and B a constant $n \times n$ matrix.

   (a) Consider the case $n = 1$. Determine B and give necessary and sufficient conditions so that every solution $x(t)$ remains bounded as $t \to \infty$.
   Hint: Introduce the average

$$\overline{A} = \frac{1}{T}\int_0^T A(s)\,\mathrm{d}s\,.$$

(b) Consider the case $n = 2$ with

$$A(t) = g(t) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where $g(t)$ is a T-periodic real-valued function. Determine B and give necessary and sufficient conditions so that every solution $x(t)$ remains bounded as $t \to \infty$.

(10+10)

3. Consider the system

$$\dot{x} = y\,,$$
$$\dot{y} = -x + y\,(1 - x^2 - 2\,y^2)\,.$$

Show that the system has at least one periodic orbit. (10)

4. Consider an autonomous planar differential equation.

(a) Suppose a point x is not on a periodic orbit. Show that a period orbit cannot be at the same time the ω-limit set and the α-limit set of the orbit through x.

(b) Show that the region bounded by a periodic orbit must contain an equilibrium point.

Hint: Argue by contradiction and use part (a).

(5+5)

5. Suppose $x(t)$ is a solution to the n-dimensional differential equation

$$\dot{x} = f(x) + \varepsilon\, g(x)$$

where $f, g \colon \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields. Further, let $y(t)$ solve the equation

$$\dot{y} = f(y)$$

with $y(0) = x(0)$.

(a) Show that $x(t) - y(t) = O(\varepsilon)$ on the time scale 1, assuming that both solutions exist on this time scale.

(b) Suppose you know that the solution $y(t)$ is bounded (therefore exists) on some time interval $[0, T]$. What can you say about the time interval of existence for $x(t)$ as $\varepsilon \to 0$?

(10+5)