Differential Equations

Homework 3

Due in class Tuesday, March 6, 2018

1. Recall that the ω-limitset of an orbit $\gamma(x)$ originating at a phase point x is the set of all accumulation points of the *forward* orbit $\gamma^+(x)$. I.e., $y \in \omega(\gamma)$ if there exists a sequence $t_n \to \infty$ such that $\phi_{t_n}(x) \to y$.

   (a) Prove that $\omega(\gamma)$ is closed and invariant.

   (b) If the forward orbit is bounded, then $\omega(\gamma)$ is compact, non-empty, and connected.

2. Recall the Poincaré–Bendixon Theorem (the proof will be finished during the next class meeting): For an autonomous differential equation in the plane, suppose a forward orbit $\gamma^+(x)$ is bounded and $\omega(\gamma)$ contains no critical points. Then $\omega(\gamma)$ is either a periodic orbit, equal to $\gamma(x)$, or a limit cycle so that $\omega(\gamma) = \overline{\gamma^+(x)}\ \gamma^+(x)$.

   In this setting, prove the following: Suppose γ_1 and γ_2 are two periodic orbits with γ_2 in the interior of γ_1. Suppose further that there are no critical points or periodic orbits in the annular region A between γ_1 and γ_2. Show that, for $x \in A$, $\omega(\gamma(x))$ is either γ_1 or γ_2. Show further that the ω-limitset is the same for all orbits in A.

3. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= a\,x_1 + b\,x_2 - x_1^2\,x_2 - x_1^3\,. \end{aligned}$$

   Show that there is cannot be a periodic orbit unless $b > 0$.

4. Show that the van der Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= -x_1 + \lambda\,(1 - x_1^2)\,x_2 \end{aligned}$$

   has a non-trivial period orbit for all $\lambda \in \mathbb{R}$.