Applied Differential Equations and Modeling

Final Exam

May 20, 2019

1. Find the inverse of the matrix
$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} .$$
(10)

2. Solve the initial value problem
$$y' - y \sin t = \sin t \,,$$
$$y(0) = \mathrm{e}^{-1} \,.$$
(10)

3. Consider the differential equation
$$y' = \mathrm{e}^{y} \,.$$
   (a) Solve the equation with initial condition $y(0) = a$.
   (b) Determine how the interval of existence depends on the initial value a.

(10)

4. Consider the system of linear differential equations
$$\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x} \,.$$
   (a) Write out the general solution.
   (b) Find the solution with
$$\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

(5+5)

5. Consider the forced harmonic oscillator described by the differential equation

$$y'' + y = g\,.$$

For $t < 0$, let $g(t) = 0$, so that $y(0) = 0$ and $y'(0) = 0$. At $t = 0$, the forcing is instantaneously increased to $g(t) = \frac{1}{2}$ and maintained at this level until, at the later time $t = c$, the forcing is instantaneously increased to its final value $g(t) = 1$.

(a) Use the Laplace transform to find the solution $y(t)$ assuming $c > 0$.

(b) How should you choose c so that the system is in a steady state (i.e., does not oscillate) for all $t > c$?

(5+5)

6. (a) Compute, without consulting the table of Laplace transforms, the Laplace transforms of the unit step function $u(t-c)$ and the delta-function $\delta(t-c)$ assuming $c > 0$.

(b) In which sense can you say that $u' = \delta$?

(5+5)

7. Consider a differential equation of the form

$$y'' + a\,y' + b\,y = g\,.$$

Show that the initial condition $y'(0) = 1$ can be replaced by adding the unit impulse function $\delta(t)$ to the forcing function g and setting $y'(0) = 0$. (5)

8. Consider the system of nonlinear differential equations

$$x' = -y\,(1-x)\,,$$
$$y' = x - y^2\,.$$

(a) Find all equilibrium points,

(b) for each equilibrium point, write out the linear system describing the evolution of small perturbations about the equilibrium points, compute its eigenvalues and, if real-valued, eigenvectors, and

(c) determine the stability of each equilibrium point and sketch the phase portrait to the extent possible.

(5+5+5)