Applied Differential Equations and Modeling

Midterm Exam

April 3, 2019

1. Solve the initial value problem

$$t\,y' + (t+1)\,y = 2\,t\,\mathrm{e}^{-t}\,,$$
$$y(1) = 1\,.$$

On which interval of time does the solution exist? (10)

2. Consider the differential equation

$$y' + y^3 = 0\,.$$

(a) Solve the equation with initial condition $y(0) = a$.

(b) Determine how the interval of existence depends on the initial value a.

(10)

3. Consider the *Gompertz growth model*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\,y\,\ln\frac{K}{y}\,,$$

where r and K are positive constants.

(a) Find the equilibrium points and classify each as stable or unstable. Sketch a few solution curves in the t-y plane.

(b) Find the general solution to the differential equation and use the formula to confirm your conclusion from part (a).

(c) Give an interpretation of the constant K when the model is used to describe population growth.

(5+5+5)

4. Find the general solution to the linear system of equations

$$
\begin{aligned}
x_1 + x_2 + 2\,x_3 &= 3\,, \\
x_1 + 2\,x_2 + x_3 - x_4 &= 4\,, \\
-x_1 - 3\,x_3 - x_4 &= -2\,.
\end{aligned}
$$

(10)

5. Consider the system of linear differential equations

$$
\boldsymbol{x}' = \begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix} \boldsymbol{x}
$$

(a) Write out the general solution,

(b) Find the solution with

$$
\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .
$$

(5+5)

6. Consider the system of nonlinear differential equations

$$
\begin{aligned}
x' &= 5\,x - x\,y\,, \\
y' &= x\,y - y\,.
\end{aligned}
$$

(a) Find all equilibrium points,

(b) for each equilibrium point, write out the linear system describing the evolution of small perturbations about the equilibrium points, compute its eigenvalues and, if real-valued, eigenvectors, and

(c) determine the stability of each equilibrium point and sketch the phase portrait to the extent possible.

(5+5+5)