Analysis II

Homework 11

Due in class Monday, May 6, 2019

1. Recall the Lagrange multiplier theorem: Let $E \subset \mathbb{R}^d$ be open, $h \in C^1(E, \mathbb{R})$, and $f \in C^1(E, \mathbb{R}^n)$ with $n < d$. Suppose that h has a local extremum subject to the constraint $f(x) = 0$ at a point $a \in E$, and suppose that $f'(a)$ has full rank. Then there exists a unique vector $\lambda \in \mathbb{R}^n$, the *Lagrange multipliers*, such that

$$\nabla h(a) = \lambda^T \, \nabla f(a) \,.$$

(Here, we think of ∇f as a row vector and λ as a column vector.)

   (a) Show that, under the assumptions of the Lagrange multiplier theorem, that if h has a local extremum subject to the constraint $f(x) = 0$ at a point $a \in E$, then the function $H \colon E \times \mathbb{R}^n \to \mathbb{R}$,

$$H(x, \mu) = h(x) - \mu^T f(x) \,,$$

   has a critical point (a, λ), where λ is the vector of Lagrange multipliers.

   (b) Give an example which shows that the critical point of H in part (a) may not correspond to a local extremum of H.

   Hint: Choose simple functions for $d = 2$ and $n = 1$, show that $\operatorname{Hess} H$ is indefinite, and appeal to the second derivative test.

2. Find the minimum and maximum values of

$$h(x, y) = x \, y$$

subject to the constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,.$$

3. (a) Calculate the iterated integral

$$\int_0^1 \int_0^1 \frac{x}{(1 + x^2)(1 + xy)} \, \mathrm{d}x \, \mathrm{d}y$$

   in two different ways, and prove thereby that

$$\int_0^1 \frac{\ln(1 + x)}{1 + x^2} \, \mathrm{d}x = \frac{\pi \ln 2}{8} \,.$$

(b) Conclude that

$$\int_0^1 \frac{\arctan x}{1+x}\,\mathrm{d}x = \frac{\pi \ln 2}{8}\,.$$

4. Consider the function

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

on the rectangle $I = [0,1] \times [\alpha, 1]$ for $\alpha \in (0,1)$.

(a) Show that

$$\int_0^1 f(x,y)\,\mathrm{d}x = -\frac{1}{1+y^2}$$

for every fixed $y \in [\alpha, 1]$.

Hint: Note that

$$\frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} + y\,\frac{\partial}{\partial y}\frac{1}{x^2+y^2}\,.$$

(b) Note that the result from (a) continuously extends to the unit square $I = [0,1]^2$ and conclude that

$$\int_0^1 \int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y = -\frac{\pi}{4}$$

while

$$\int_0^1 \int_0^1 f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \frac{\pi}{4}\,.$$

(c) Does this contradict the theorem on the exchange of partial integration proved in class? Explain!