Analysis II

Homework 12

Due in class Monday, May 13, 2019

1. (Kantorovitz, Exercise 4.2.15.1) Let D be the domain bounded by the parabola $x = y^2$ and the line $x = y$. Calculate the integral
$$\int_D \sin \frac{\pi x}{y} \, \mathrm{d}S \, .$$

2. (Kantorovitz, Exercise 4.2.15.2) Let D be the annulus bounded by circles centered at the origin with radii $0 < a < b$. Compute
$$\int_D \arctan \frac{y}{x} \, \mathrm{d}S \, .$$

3. (Kantorovitz, Exercise 4.2.15.6) Find the area of the closed domain bounded by the two parabolas $y^2 = ax$ and $y^2 = bx$ ($0 < a < b$), and the two hyperbolas $xy = \alpha$ and $xy = \beta$ ($0 < \alpha < \beta$).

   Hint: Use the map
$$(x, y) \mapsto (u, v) \equiv (y^2/x, xy) \, .$$

4. Let D be a domain in $\mathbb{R}^n$. A function $f \in C^2(D)$ is called *harmonic* if $\Delta f = 0$, where Δ is the *Laplace operator* defined via
$$\Delta f = \nabla \cdot \nabla f = \partial_1^2 f + \cdots + \partial_n^2 f \, .$$
   Show that a harmonic function has the mean value property
$$f(x) = \frac{1}{S(\partial B(x,r))} \int_{\partial B(x,r)} f \, \mathrm{d}S$$
   for every $x \in D$ and every $r > 0$ sufficiently small such that $\partial B(x,r) \subset D$, where $B(x,r)$ denotes the ball centered at x with radius r and
$$S(\partial B(x,r)) = \int_{\partial B(x,r)} \mathrm{d}S$$
   is the $(n-1)$-dimensional content of $\partial B(x,r)$.

   Hint: Proceed in the following steps:

(a) Define

$$\phi(r) = \frac{\displaystyle\int_{\partial B(x,r)} f \, \mathrm{d}S}{\displaystyle\int_{\partial B(x,r)} \mathrm{d}S}.$$

Now use a change of variables in numerator and denominator which takes $B(x, r)$ to $B(0, 1)$.

(b) Differentiate with respect to r, move the differentiation under the integral, and apply the chain rule.

(c) Apply the divergence theorem and use that f is harmonic to conclude that $\phi'(r) = 0$.

(d) Finish the proof by considering the limit $r \to 0$.