Analysis II

Homework 2

Due in class Monday, February 18, 2019

1. Suppose $f = a\,(x-1)^2 + \psi(x)\,(x-1)^3$ satisfies the requirements of the simple version of Laplace method from class (also see the handout for a precise statement). Suppose g is smooth, bounded, with $g(1) > 0$. Show that, as $s \to \infty$,

$$\int_0^\infty \mathrm{e}^{-sf(x)}\, g(x)\, \mathrm{d}x \sim g(1) \sqrt{\frac{\pi}{as}}\,.$$

   Hint: Set

$$G(x) = \int_1^x g(z)\, \mathrm{d}z\,,$$

   then change variables $y = G(x)$ in the first integral.

2. Suppose $f \colon [0, \infty) \to \mathbb{R}$ is continuous and

   (i) $f(0) > 0$,

   (ii) there exists $\delta, M > 0$ such that f is differentiable on $[0, \delta]$ with $|f'(x)| \le M$ for every $x \in [0, \delta]$,

   (iii) there exist $b, C > 0$ such that $|f(x)| \le C\,\mathrm{e}^{bx}$ for all $x \ge 0$.

   In this setting,

   (a) Show that, as $s \to \infty$,

$$\int_0^\infty \mathrm{e}^{-sx}\, f(x)\, \mathrm{d}x \sim \frac{f(0)}{s}\,,$$

   (b) Formulate a theorem which provides higher order corrections to the statement from (a); you will need to adjust your assumptions on f accordingly.

   (c) Test your formula from part (b) on the integral

$$\int_0^\infty \mathrm{e}^{-sx}\, (1 - \cos x)\, \mathrm{d}x\,.$$

   (Note: for this integral, you can find the leading order asymptotics by direct repeated integration by parts to compare.)

3. (Rudin, Theorem 7.9.) Let $f, f_n \colon E \to \mathbb{R}$ with pointwise limit

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every $x \in E$. Set

$$M_n = \sup_{x\in E} |f_n(x) - f(x)| \,.$$

Show that $f_n \to f$ uniformly in E if and only if $\lim_{n\to\infty} M_n = 0$.

4. (Rudin, Example 7.4.) Let

$$f_n(x) = \lim_{m\to\infty} (\cos n!\pi x)^{2m} \,.$$

   (a) Argue that

$$f_n(x) = \begin{cases} 1 & \text{if } n!x \in \mathbb{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

   (b) Show that the pointwise limit

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{I}\,, \\ 1 & \text{if } x \in \mathbb{Q}\,. \end{cases}$$

   (c) Argue that f is not Riemann integrable.

5. (Rudin, Exercise 7.9.) Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$.