Analysis II

Homework 3

Due in class Monday, February 25, 2019

1. Let X be a metric space and $E \subset X$. We say that $x \in X$ is a *boundary point* of E if every neighborhood of x contains at least one point in E and at least one point in E^c. The set of all boundary points of E is denoted ∂E. Prove the following.

   (a) $E \setminus \partial E$ is open.

   (b) $E \cup \partial E$ is closed.

   (c) ∂E is closed.

2. Let X be a set. Define, for all $x, y \in X$, the "trivial metric"

$$d(x,y) = \begin{cases} 0 & \text{if } x = y\,, \\ 1 & \text{if } x \neq y\,. \end{cases}$$

   (a) Show that d is indeed a metric.

   (b) Show that in this metric space, every subset $E \subset X$ is both open and closed.

3. Show that the union of a finite number of compact subsets of a metric space X is compact.

4. Let X be a metric space and $K \subset X$ compact. Let $\{G_\alpha\}$ be an open cover of K. Show that there exists $\lambda > 0$ such that for every $E \subset K$ with $\operatorname{diam} E \leq \lambda$ there exists an index α such that $E \subset G_\alpha$.

5. Let $E \subset \mathbb{R}$ be bounded and $f \colon E \to \mathbb{R}$ be uniformly continuous. Show that f is bounded on E.

6. Let X, Y be metric spaces and $f \colon X \to Y$ continuous. Show that $f(K)$ is compact if K is compact.