Analysis II

Homework 8

Due in class Monday, April 8, 2019

1. Let V and W be normed vector spaces and $f\colon V \to W$ be continuously differentiable and homogeneous of degree $\alpha > 0$, i.e., for every $x \in V$ and $t > 0$,

$$f(tx) = t^\alpha f(x)\,.$$

Show that

$$f'(x)x = \alpha\, f(x)\,.$$

2. Let V be a normed vector space, $E \subset V$ open, and $f\colon E \to \mathbb{R}$ differentiable on E. Suppose that f has a local maximum at some point $x \in E$. Show that, for every $\boldsymbol{v} \in V$,

$$D_{\boldsymbol{v}} f(x) = 0\,.$$

Remark: If $V \in \mathbb{R}^n$, this implies that

$$\nabla f(x) \equiv (\partial_1 f(x), \ldots, \partial_n f(x)) = 0\,.$$

3. Let X be a metric space and $A, B \subset X$. We say that A and B are *separated* if

$$A \cap \overline{B} = \emptyset = \overline{A} \cap B\,.$$

Moreover, $C \subset X$ is said to be *connected* if it is not the union of two non-empty separated sets.

   (a) Show that $\overline{C}$ is connected if C is connected.

   (b) Give an example that the interior of a connected set may not be connected.

4. Let X be a metric space and $a, b \in X$. We say that $\gamma\colon [0,1] \to X$ is a *path from a to b* if γ is continuous with $f(0) = a$ and $f(1) = b$.

   (a) Let $E \subset X$ be open, fix $a \in E$, and define

$$\Gamma = \{x \in E\colon \text{there exists a path from } a \text{ to } x\}\,.$$

Show that Γ is open and closed in E.

Note: Here the notion of open and closed are *relative* to E, i.e., we consider E itself as a metric space with the metric inherited from X. E.g., if $X \in \mathbb{R}^2$ and E is the open unit disk centered at the origin, then

$$F = \{x \in E \colon x_1, x_2 \geq 0\}$$

is closed in E, even though it is clearly not closed in X.

(b) We say that $E \subset X$ is *path-connected* if for every $a, b \in E$ there exists a path from a to b. Use part (a) to argue that for open sets, the notion of connectedness and path-connectedness is equivalent.

5. Let $E \subset \mathbb{R}^n$ be open and connecte and $f \colon E \to \mathbb{R}^m$ be differentiable. Show that if $f'(x) = 0$ for every $x \in E$, then f is constant in E.

6. Let V and W be normed vector spaces and let $A \colon V \to W$ and $B \colon W \to V$ be bounded linear operators. Furthermore, suppose that

$$\|I - BA\|_{L(V)} < 1\,.$$

(a) Show that $F \colon L(W, V) \to L(W, V)$, defined for every $X \in L(W, V)$ by

$$FX = X + B - BAX\,,$$

is a contraction.

(b) Conclude that F has a fixed point X^*. State an upper bound for the operator norm $\|X^*\|_{L(W,V)}$.