Final Exam Solutions

1. (a) By the inverse function theorem, f is a local diffeomorphism, i.e. $\forall x \in A$ $\exists$ open sets $U_x \subset A$ and $V_x \subset B = f(A)$ s.t. $f : U_x \to V_x$ is a bijection.

Since
$$B = \{ f(x) : x \in A \} \subset \bigcup_{x \in A} V_x \subset B,$$

we have $B = \bigcup_{x \in A} V_x$ so that B is open as a union of open sets.

(b) Take $A = \mathbb{R}$, $f(x) = x^2$

$\Rightarrow B = f(A) = [0, \infty)$ is NOT an open set.

Correspondingly, $f'(x) = 2x$, so $f'(0)$ is not invertible.

2. (a) Let $u(t)$ denote the curve in $\mathcal{A}$ given by

$$u(t) = u + tv \quad , \; u, v \in \mathcal{A}$$

and, as usual, define $\delta F = \frac{d}{dt} F(u(t)) \Big|_{t=0}$ for any map F defined on $\mathcal{A}$.

So
$$\delta E \overset{(*)}{=} \underbrace{\int_D \nabla \delta u \cdot \nabla u \, dx}_{} + \int_D \delta u \, f \, dx$$

$$= \int_{\partial D} \delta u \, \underbrace{\hat{n} \cdot \nabla u}_{=0 \text{ on } \partial D} \, d\sigma - \int_D \delta u \, \Delta u \, dx \quad \text{by divergence theorem}$$

$$= \int_D \delta u \, (-\Delta u + f) \, dx$$

Since $\delta E = dE(u)\,\delta u$, identifying $\delta u \equiv v$, we see that

$$dE(u)\,v = \int_D v \, (-\Delta u + f) \, dx$$

Note: the interchange of differentiation and the integral is possible because the t-derivative of the integrand is uniformly continuous on $\overline{D}$.

(b) At a local minimum, $dE(u)v = 0$ for every $v \in \mathcal{A}$.

Since $-\Delta u = f$ is continuous, this implies

$$\Delta u = f \quad \text{in } D.$$

The boundary condition $u = 0$ on ∂D is already "baked into" the definition of the domain $\mathcal{A}$.

3. Use Lagrange multipliers:

$$\nabla f = 2x$$

$$\nabla g_1 = \nabla(x_1 + 2x_2 + x_3 - 1) = (1, 2, 1, 0, 0)^T$$

$$\nabla g_2 = \nabla(x_3 - 2x_4 + x_5 - 6) = (0, 0, 1, -2, 1)^T$$

Need $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$$\Rightarrow 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \quad g_1 = 0, \quad g_2 = 0$$

This is a system of linear equations:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 6 \end{pmatrix}$$

Now do Gaussian elimination on the augmented matrix:

$$\left(\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{35}{12} & \frac{35}{6} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{c|c} I & \begin{matrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \\ 0 \\ 2 \end{matrix} \end{array}\right)$$

$\Rightarrow$ the unique critical point was

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{with} \quad \|x\|^2 = 6 .$$

Since f is convex, this is the unique minimum.

4. $$e^t - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!}$$

$$\Rightarrow \frac{e^t - 1}{t} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$$

Since e^t is absolutely convergent on $\mathbb{R}$, $\frac{e^t-1}{t}$ is also

absolutely convergent on $\mathbb{R}$ (you can also easily show this

via the ratio test), so that we can integrate term-by-term, with

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2 \, n!} .$$

5. $$\overline{f_k} = \frac{1}{2\pi} \int_0^{2\pi} \overline{e^{-ikx} f(x)} \, dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(-k)x} f(x) \, dx = f_{-k} \quad (*)$$

Similarly,

$$\overline{f_k} = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik(-x)} f(x) \, dx \qquad x = -y$$

$$= \frac{1}{2\pi} \int_0^{-2\pi} e^{-iky} f(-y) \, (-dy)$$

$$= -\frac{1}{2\pi} \int_{-2\pi}^{0} e^{-iky} f(y) \, dy$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} e^{-iky} f(y)\,dy \qquad \text{by periodicity}$$

$$= -f_k$$

This proves that f_k is purely imaginary; moreover, with $(*)$, $f_{-k} = -f_k$.

6. Clearly, $(u,v) \in U = [-1,1]\times[-1,1]$

$$(x,y) = \Phi(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

$$D\Phi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad \Rightarrow \left|\det D\Phi\right| = \frac{1}{2}$$

$$\Rightarrow \int_D \frac{(x-y)^2}{(x+y+2)^2}\,dx\,dy = \int_U \frac{v^2}{(2+u)^2}\left|\det D\Phi\right|\,du\,dv$$

$$= \underbrace{\int_{-1}^{1} v^2\,dv}_{=\frac{1}{3}v^3\Big|_{-1}^{1}} \; \underbrace{\int_{-1}^{1}\frac{1}{(2+u)^2}\,du}_{\frac{-1}{2+u}\Big|_{-1}^{1}} \cdot \frac{1}{2}$$

$$= \frac{2}{3}\cdot\left(1-\frac{1}{3}\right)\cdot\frac{1}{2}$$

$$= \frac{2}{9}$$

7. "$\Rightarrow$": F conservative $\Rightarrow F = \nabla\phi$ for some $\phi \in C^2(D)$

$$\Rightarrow \nabla\times F = \nabla\times\nabla\phi = 0$$

"$\Leftarrow$": Let γ be a simple closed curve. Take a "capping surface" M s.t. $\gamma = \partial M$ (proof of existence is subtle - here we assume existence without further argument). Then

$$\int_\gamma F \cdot dx = \int_M \underbrace{(\nabla \times F)}_{=0} \cdot \hat{n} \, d\sigma \qquad \text{(Stokes' theorem)}$$

$$= 0$$

$\Rightarrow F$ is conservative.

8. (a) $$\nabla^\perp \cdot F = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$= 0$$

$\Rightarrow F$ is conservative (By Q7 restricted to 2-D, or a direct application of Green's theorem)

(b) When γ does not enclose the origin, the line integral

$$\int_\gamma F \cdot dx = 0 \quad \text{by (a)}$$

When γ encloses the origin, it suffices to consider the integral along the unit circle centered at the origin, which we parametrize by standard polar coordinates:

$$\gamma = (\cos\theta, \sin\theta)$$

$$\Rightarrow \gamma' = (-\sin\theta, \cos\theta)$$

$$\Rightarrow \int_\gamma F \cdot dx = \int_0^{2\pi} (-\sin\theta, \cos\theta) \cdot (-\sin\theta, \cos\theta) \, d\theta = 2\pi$$

(c) Again, if γ does not enclose the origin, then $\int_\gamma \bar{z}\, dz = 0$

by Cauchy's theorem.

Otherwise, $\int_\gamma \frac{1}{z}\, dz = 2\pi i \underbrace{\mathrm{Res}\left(\frac{1}{z}, 0\right)}_{=1} = 2\pi i$

(d) Set $z = x + iy$

$\Rightarrow \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \equiv u + iv$ with $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

$$\int_\gamma \frac{1}{z}\, dz = \int_\gamma (u+iv)(dx + i dy) = \int_\gamma u\,dx - v\,dy + i \underbrace{\int_\gamma u\,dy + v\,dx}_{\equiv \int_\gamma F \cdot dx}$$

as in (b)

This must be zero, as follows
by comparison with (c)
or by direct calculation using the parameterization from (b).

9. (a) For γ_1: $z = 0 \Rightarrow x^2 + y^2 = 4$

Parameterize via polar coordinates: $\gamma_1 = 2(\cos\theta, \sin\theta, 0)$

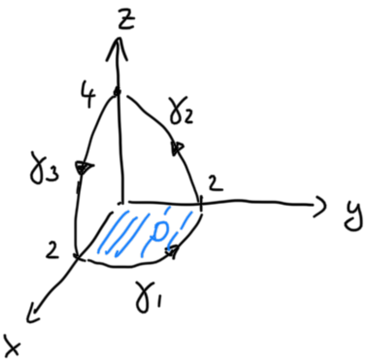
$\Rightarrow \gamma_1' = 2(-\sin\theta, \cos\theta, 0)$

$$\int_{\gamma_1} F \cdot dx = \int_0^{\pi/2} \underbrace{(0,0,1) \cdot \gamma_1'}_{=0}\, d\theta = 0$$

For γ_2: $x = 0$, use $\gamma_2 = \gamma_2(y) = (0, y, 4 - y^2)$, $y \in [0, 2]$

$\gamma_2' = (0, 1, -2y)$

Define orientation:



$$\Rightarrow \int_{\gamma_2} F \cdot dx = \int_2^0 \underbrace{(y(4-y^2), 0, 1) \cdot (0, 1, -2y)}_{= -2y} dy$$

$$= 2 \int_0^2 y \, dy = y^2 \Big|_0^2 = 4$$

For γ_3: $y = 0$, use $\gamma_3 = \gamma_3(x) = (x, 0, 4-x^2)$

$$\Rightarrow \gamma_3' = (1, 0, -2x)$$

$$\int_{\gamma_3} F \cdot dx = \int_0^2 \underbrace{(0, -x(4-x^2), 1) \cdot (1, 0, -2x)}_{= -2x} dx$$

$$= -2 \int_0^2 x \, dx = -x^2 \Big|_0^2 = -4$$

$$\Rightarrow \int_{\gamma} F \cdot dx = 0 + 4 - 4 = 0$$

(b) M is parameterized by $f(x,y) = (x, y, 4 - x^2 - y^2)$

$$\Rightarrow \frac{\partial f}{\partial x} = (1, 0, -2x)$$

$$\frac{\partial f}{\partial y} = (0, 1, -2y)$$

$$\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y} = (2x, 2y, 1)$$

$$\Rightarrow \quad n = \frac{\partial}{\partial x} \times \frac{\partial}{\partial y} = (2x, 2y, 1)$$

$$\nabla \times F = \big(0 - (-x),\ y - 0,\ -z - z\big) = (x, y, -2z)$$

$$\int_\gamma F \cdot dx = \int_M (\nabla \times F) \cdot \hat{n} \, d\sigma = \int_D (\nabla \times F) \cdot n \; d(x,y)$$

where D is the disk of radius 2 restricted to the first quadrant of the x-y plane, see sketch.

So we need to compute

$$\int_D \underbrace{(x, y, -2(4 - x^2 - y^2)) \cdot (2x, 2y, 1)}_{= 4x^2 + 4y^2 - 8} \; d(x,y)$$

$$= \int_0^2 \int_0^{\pi/2} (4r^2 - 8)\, r \, d\theta \, dr$$

$$= \frac{\pi}{2} \left(4 \tfrac{1}{4} r^4 - 8 \tfrac{1}{2} r^2 \right) \Big|_0^2 = \frac{\pi}{2} \left(16 - 4 \cdot 4 \right) = 0$$

Which coincides with the answer from part (a).