Advanced Calculus and Methods of Mathematical Physics

Final Exam

May 25, 9:00–11:00

1. Let $A \subset \mathbb{R}^n$ be open, $f \in C^1(A, \mathbb{R}^n)$, and suppose its Jacobian matrix $DF(x)$ is invertible for every $x \in A$.

   (a) Show that $B = f(A)$ is open, even if f is not injective.

   (b) Give an example which shows that the assumption on the Jacobian is necessary.

(5+5)

2. Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let

$$\mathcal{A} = \{u \in C^2(\overline{D}, \mathbb{R}) \colon u = 0 \text{ on } \partial D\}$$

and define a function $E \colon \mathcal{A} \to \mathbb{R}$ by

$$E(u) = \int_D \bigl(\tfrac{1}{2} \nabla u \cdot \nabla u + u\, f\bigr) \, \mathrm{d}x$$

for some given function $f \in C(\overline{D}, \mathbb{R})$.

   (a) Show that the derivative of E at a point $u \in \mathcal{A}$ is the linear map acting on arbitrary $v \in \mathcal{A}$ via

$$\mathrm{d}E(u)v = \int_D \bigl(-v\, \Delta u + v\, f\bigr) \, \mathrm{d}x \, .$$

   Hint: Divergence theorem. And don't forget to argue why it is possible to interchange integration and differentiation.

   (b) Use the result from part (a) to show that if E has a local minimum at some $u \in \mathcal{A}$, then u solves Poisson's equation

$$\begin{aligned} \Delta u &= f \quad \text{in } D \, , \\ u &= 0 \quad \text{on } \partial D \, . \end{aligned}$$

(5+5)

3. Minimize $f \colon \mathbb{R}^5 \to \mathbb{R}$,

$$f(x) = \|x\|^2$$

subject to

$$x_1 + 2\,x_2 + x_3 = 1\,,$$
$$x_3 - 2\,x_4 + x_5 = 6\,.$$

(10)

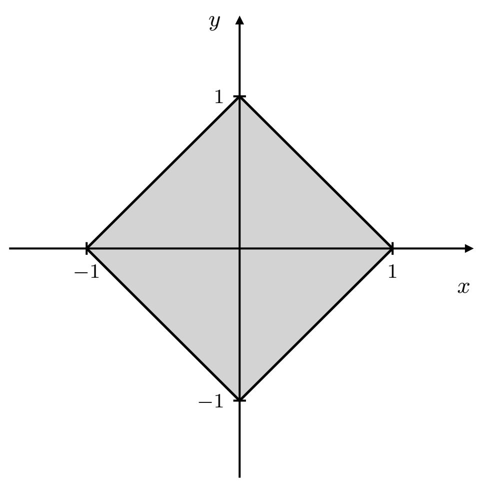
4. Find the power series expansion for the function

$$f(x) = \int_0^x \frac{\mathrm{e}^t - 1}{t}\,\mathrm{d}t$$

and determine its radius of convergence. (5)

5. Let f be an odd (i.e., $f(x) = -f(-x)$), real-valued 2π-periodic function. Show that its Fourier coefficients are also odd (i.e., $f_k = -f_{-k}$) and purely imaginary. (5)

6. Let $D \subset \mathbb{R}^2$ denote the diamond-shaped shaded region sketched below.



Compute

$$\int_D \frac{(x-y)^2}{(x+y+2)^2}\,\mathrm{d}x\,\mathrm{d}y\,.$$

Hint: use the change of variables $u = x + y$, $v = x - y$. (10)

7. Let $D \subset \mathbb{R}^3$ be a simply connected domain. Show that a vector field $F \in C^1(D, \mathbb{R}^3)$ is conservative if and only if $\nabla \times F = 0$. (5+5)

8. Let

$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

be a vector field defined on $D = \mathbb{R}^2 \setminus \{0\}$.

(a) Show that F is locally conservative.

(b) Compute the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

(i) for any simple closed curve γ encircling the origin, (ii) for any simple closed curve γ *not* encircling the origin.

(c) Set $z = x + \mathrm{i}y$. Use the residue theorem to compute

$$\int_\gamma \frac{1}{z} \, \mathrm{d}z$$

for any simple closed curve γ encircling the origin in the complex plane, (ii) for any simple closed curve γ *not* encircling the origin.

(d) State an identification between the real-variable computation from (b) and the complex-variable computation from (c).

(5+5+5+5)

9. Let

$$F(x,y,z) = (y\,z, -x\,z, 1)$$

be a vector field in $\mathbb{R}^3$. Let M be given by the surface of the paraboloid

$$z = 4 - x^2 - y^2$$

restricted to the first octant. Let γ_1, γ_2, and γ_3 be the part of the boundary of M lying in the xy, yz, and xz-planes, respectively and set $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, in standard orientation.

(a) Compute the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

directly as line integrals along γ_1, γ_2 and γ_3 with suitable parameterizations.

(b) Compute the same line integral via Stokes' theorem as a surface integral over the capping surface M.

(10+10)