Homework 10 Solutions

1 (a). $$\overline{f_k} = \overline{\frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\,dx}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{e^{ikx}}_{=e^{-i(-k)x}} \underbrace{\overline{f}(x)}_{= f(x) \text{ as } f \text{ is real-valued}}\,dx$$

$$= f_{-k}$$

(b) $$f'(x) = \frac{d}{dx}\sum_{k\in\mathbb{Z}} f_k\, e^{ikx}$$

$$= \sum_{k\in\mathbb{Z}} \underbrace{f_k\, ik}_{=(f')_k}\, e^{ikx}$$

(c) $$(T_a f)_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x-a)\,dx$$

$y = x - a$
$\Rightarrow dx = dy$

$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{e^{-ik(y+a)}}_{=e^{-iky}e^{-ika}} f(y)\,dy$$

$$= e^{-ika} f_k$$

(d) $$(f*g)(x) = \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f_k\, e^{iky} \sum_{l\in\mathbb{Z}} g_l\, e^{il(x-y)}\,dy$$

$$= \sum_{k,l\in\mathbb{Z}} f_k g_l \int_0^{2\pi} \underbrace{e^{iky}\, e^{il(x-y)}}_{=e^{ilx}e^{i(k-l)y}}\,dy$$

$$= e^{ilx} \underbrace{\int_0^{2\pi} e^{i(k-l)y}\, dy}_{= 2\pi \delta_{kl}}$$

$$= 2\pi \sum_{k\in\mathbb{Z}} f_k\, g_k\, e^{ikx}$$

$$\Rightarrow (f*g)_k = 2\pi\, f_k\, g_k .$$

2. $$f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\, x\, dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-ikx}}{-ik}\, x \,\Bigg|_{x=-\pi}^{x=\pi} - \frac{1}{ik} \underbrace{\int_{-\pi}^{\pi} e^{-ikx}\, dx}_{=0} \right] \qquad \text{for } k\neq 0$$

$$= \frac{1}{2\pi}\, \frac{(-1)^k \pi - (-1)^k(-\pi)}{-ik} = \frac{i}{k}(-1)^k$$

3(a): f is a polynomial in $z \Rightarrow f$ analytic $\Rightarrow f$ holomorphic

(We can also check the CR equations explicitly:

$$z^2 = (x+iy)^2 = x^2 + 2ixy - y^2$$

$$\Rightarrow u = x^2 - y^2, \quad v = 2xy \qquad \text{where } f = u + iv$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} . \;)$$

(b) $|z|^2 = (x+iy)(x-iy) = x^2 + y^2 \quad \Rightarrow f = u + iv$ with $u = x^2+y^2$, $v=0$

(b) $|z|^2 = (x+iy)(x-iy) = x^2+y^2 \; \ldots \; + \; \ldots \; 0.$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x \neq 0 = \frac{\partial v}{\partial y}$$

$\Rightarrow$ C-R equations are not satisfied, hence f is not holomorphic

(c) $f(z) = \cos z$ has a Taylor series that converges everywhere on $\mathbb{C}$ (e.g. use ratio-test on the explicit series)

$\Rightarrow f$ analytic $\Rightarrow f$ holomorphic.

4. By assumption, f satisfies the C-R. equations, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A similar calculation by cross-differentiation proves that $\Delta v = 0$.

Note: this argument is rigorous as f holomorphic implies that $f \in C^\infty$, so in particular the second-order partial derivatives of u and v exist.

5. The only pole of $f(z) = z^2 \sin \frac{1}{z}$ is at $z=0$, inside the contour of integration. Since

$$z^2 \sin \frac{1}{z} = z^2 \left(z - \frac{1}{3!} z^3 + O(z^5) \right),$$

$$\mathrm{Res}(f,0) = -\frac{1}{3!} = -\frac{1}{6}, \text{ so}$$

$$\int_{|z|=1} f(z)\,dz = 2\pi i \,\mathrm{Res}(f,0) = -\frac{\pi i}{3}$$

6 (a).

$$\phi(r) = \frac{\int_{\partial B(x,r)} f(y)\,dS(y)}{\int_{\partial B(x,r)} dS}$$

write $y = x + r\xi \equiv \Phi(\xi)$

with $\|\xi\| = 1$

Note: $\det|D\Phi| = r^{n-1}$

$$= \frac{\int_{B(0,1)} f(x+r\xi)\, r^{n-1}\,dS(\xi)}{\int_{\partial B(0,1)} r^{n-1}\,dS}$$

$$= \frac{1}{S(\partial B(0,1))}\int_{\partial B(0,1)} f(x+r\xi)\,dS(\xi)$$

(b) $$\phi'(r) = \frac{1}{S(\partial B(0,1))}\int_{\partial B(0,1)} \nabla f(x+r\xi)\cdot\xi\,dS(\xi)$$

(by Leibnitz' rule as $f \in C^1(D)$)

(c) Note that $n = \xi$ is the outward unit vector on $\partial B(0,1)$, so

$$\phi'(r) = \frac{1}{S(\partial B(0,1))}\int_{\partial B(0,1)} n\cdot\nabla f(x+r\xi)\,dS(\xi)$$

$$= \frac{1}{S(\partial B(0,1))} \int_{B(0,1)} \nabla_\xi \cdot \nabla_y f(x+r\xi)\, d\xi \qquad \text{(divergence theorem)}$$

Note: ∇_y is the gradient with respect to the argument of f, while ∇_ξ is the gradient with respect to the variable of integration.

Since $y = x + r\xi$, $\nabla_\xi = r\nabla_y$, so that

$$\nabla_\xi \cdot \nabla_y f(x+r\xi) = r\left(\nabla_y \cdot \nabla_y f\right)(x+r\xi)$$

$$= r\,\Delta f(x+r\xi)$$

$$= 0 \quad \text{since } f \text{ is harmonic.}$$

$$\Rightarrow \phi'(r) = 0$$

(d) Since f is cont., $\phi(0) \equiv \lim_{r\to 0} \phi(r) = \dfrac{f(x)\int_{\partial B(x,r)} dS}{\int_{\partial B(x,r)} dS} = f(x)$

Since $\phi(r) = \phi(0) + \int_0^r \underbrace{\phi'(r)}_{=0}\, dr$,

$$\frac{\int_{\partial B(x,r)} f(y)\, dS(y)}{\int_{\partial B(x,r)} dS(y)} = f(x)$$

<u>Note:</u> This is a real-variable version of the Cauchy

integral formula for holomorphic functions. It's valid in any dimension n, but requires integration on surfaces of balls, while the Cauchy integral formula in $\mathbb{C}$ holds for arbitrary curves encircling the center point.