Partial Integrals

$I = [a,b] \times [\alpha,\beta] = I_1 \times I_2$

$f : I \to \mathbb{R}$; $f(\cdot, y)$ shall be R-integrable for every $y \in I_2$

$$F(y) = \int_a^b f(x,y)\,dx \qquad F: I_2 \to \mathbb{R} \quad \text{"partial integral"}$$

Interlude: Uniform continuity

__Def.__: X, Y metric spaces, $f: X \to Y$ unif. cont. if

$\forall \varepsilon > 0 \ \exists \delta$ s.t. $\forall x, x' \in X$ with $d(x,x') < \delta$: $d(f(x), f(x')) < \varepsilon$.

__Theorem__: K compact, $f: K \to Y$ cont. $\Rightarrow$ f unif. cont.

__Theorem__: K compact, $f: K \to Y$ $\underbrace{\text{cont.}}_{(**)}$ $\Rightarrow$ f unif. cont.

__Proof__: Suppose the contrary: $\exists \varepsilon > 0$ s.t. for $\delta = \frac{1}{n}$ there are

$x_n, x_n' \in K$ with $\underbrace{d(x_n, x_n') < \frac{1}{n}}_{(*)}$ and $\underbrace{d(f(x_n), f(x_n')) \geq \varepsilon}_{(***)}$.

K compact, so x_n, x_n' have converging subsequences, for convenience, continue to write x_n, x_n'

$$\left.\begin{array}{l} x_n \to x \in K \\ x_n' \to x' \in K \end{array}\right\} \xRightarrow{(*)} x = x'$$

By $(**)$, $\left.\begin{array}{l} f(x_n) \to f(x) \\ f(x_n') \to f(x') \end{array}\right\}$ same limit

This contradicts $(***)$ $\square$

Theorem: $f \in C(I) \Rightarrow F \in C(I_2)$

Proof: Let $\varepsilon > 0$. Since f unif. cont. on the compact set I, $\exists \delta$ s.t. $\forall x,\ y, y'$ with $|y - y'| < \delta$

$$|f(x,y) - f(x,y')| < \frac{\varepsilon}{b-a}$$

$$|F(y) - F(y')| = \left| \int_a^b \big(f(x,y) - f(x,y')\big)\, dx \right| \leq \int_a^b \underbrace{|f(x,y) - f(x,y')|}_{\frac{\varepsilon}{b-a}}\, dx$$

$$\leq \varepsilon$$

$\square$

Theorem (Leibniz' rule I): $f \in C(I)$, $\frac{\partial f}{\partial y} = f_y \in C(I)$

$\Rightarrow F \in C^1(I_2)$ with

$$F'(y) = \int_a^b f_y(x,y)\, dx$$

Proof: Let $\varepsilon > 0$. Since f_y unif. cont. on I, $\exists \delta$ s.t. $\forall |\theta| < \delta$:

$$|f_y(x, y+\theta) - f_y(x,y)| < \frac{\varepsilon}{b-a} \qquad \forall (x,y) \in I,\ (x, y+\theta) \in I$$

$$\left| \frac{F(y+h) - F(y)}{h} - \int_a^b f_y(x,y)\, dx \right| = \left| \int_a^b \Big(\underbrace{\frac{f(x,y+h) - f(x,y)}{h}}_{\overset{MVT}{=}\ f_y(x,\, y+\theta(x,h))} - f_y(x,y) \Big)\, dx \right|$$

$|\theta| \leq |h|$

$$< (b-a) \frac{\varepsilon}{b-a} = \varepsilon$$

$\square$

Theorem (Leibnitz rule II): Let $I_1 = [a, \infty)$, $I_2 = [\alpha, \beta]$ $\quad I = I_1 \times I_2$

$f, f_y \in C(I)$ with

(i) $F(y)$ finite $\forall y \in I_2$

(ii) $\int_a^\infty f_y(x,y)\,dx$ converges absolutely and uniformly on I_2

(I.e: $\forall \varepsilon > 0 \ \exists b$ s.t. $\int_b^\infty |f_y(x,y)|\,dy < \varepsilon \quad \forall y \in I_2$)

$\Rightarrow F \in C^1(I_2)$ with

$$F'(y) = \int_a^\infty f_y(x,y)\,dx$$

Proof: Let $\varepsilon > 0$, following the previous proof,

$$\left| \frac{F(y+h) - F(y)}{h} - \int_a^\infty f_y(x,y)\,dy \right| \le \int_a^\infty | f_y(x, y + \theta(x,h)) - f_y(x,y) |\,dx$$

$$\le \underbrace{\int_a^b | f_y(x, y+\theta(x,h)) - f_y(x,y) |\,dx}_{\le \frac{\varepsilon}{(b-a)3}} + \underbrace{\int_b^\infty | f_y(x, y+\theta(x,h)) |\,dx}_{< \frac{\varepsilon}{3}} + \underbrace{\int_b^\infty |f_y(x,y)|\,dx}_{\le \frac{\varepsilon}{3}}$$

as in previous proof, in step 3, since b is already chosen, choose δ s.t. $\le \frac{\varepsilon}{(b-a)3}$

by same argument (with same b) 2

$\le \frac{\varepsilon}{3}$ for 1 b large enough by (ii)

$\le \varepsilon$ □

Example: $f(x,y) = e^{-xy} \frac{\sin x}{x}$ on $I = [0,\infty) \times [\alpha, \beta]$, $0 < \alpha < \beta$

$$f_y = -e^{-xy} \sin x$$

Since $y \geq \alpha > 0$, the integral $\int_0^\infty e^{-xy} \sin x \, dx$ is unif. conv.

$$\underset{\substack{\uparrow \\ \text{Theorem}}}{F'(y) =} - \int_0^\infty e^{-xy} \sin x \, dx = \underbrace{e^{-xy} \cos x \Big|_{x=0}^{x=\infty}}_{=-1} - (-y) \int_0^\infty e^{-xy} \cos x \, dx$$

$$= -1 + y e^{-xy} \sin x \Big|_{x=0}^{x=\infty} - (-y) y \underbrace{\int_0^\infty e^{-xy} \sin x \, dx}_{=-F'(y)}$$

$$\Rightarrow F'(y) = -1 - y^2 F'(y) \Rightarrow F'(y) = -\frac{1}{1+y^2}$$

$$\Rightarrow F(\beta) - F(\alpha) = \arctan \alpha - \arctan \beta$$

Take $\beta \to \infty$: $0 - F(\alpha) = \arctan \alpha - \frac{\pi}{2}$

Or: $F(\alpha) = \frac{\pi}{2} - \arctan \alpha$

$$F(\alpha) = \int_0^\infty e^{-\alpha x} \frac{\sin x}{x} \, dx$$

Let $\alpha \to 0$, then formally, $F(\alpha) \to \int_0^\infty \frac{\sin x}{x} \, dx$

$$\Rightarrow \int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$$ "Dirichlet Integral"

"Feynman's trick"

Theorem (Leibnitz rule III) $\quad I = \underbrace{[a,b]}_{I_1} \times \underbrace{[\alpha,\beta]}_{I_2}$

$$f, f_y \in C(I)$$

$$\phi, \psi \in C'(I_2, I_1)$$

$$H(y) = \int_{\phi(y)}^{\psi(y)} f(x,y)\, dy$$

$\Rightarrow \quad H \in C'(I)$ with

$$H(y) = \int_{\phi(y)}^{\psi(y)} f_y(x,y)\, dy + f(\psi(y),y)\,\psi'(y) - f(\phi(y),y)\,\phi'(y)$$

Proof: Define

$$\left.\begin{array}{l} F(y,u,v) = \displaystyle\int_u^v f(x,y)\, dx \\[2em] G(y) = (y, \phi(y), \psi(y)) \end{array}\right\} \quad H = F \circ G$$

Fact: For fixed u,v, F satisfies the requirements of Leibnitz I,

$$F_y = \int_u^v f_y(x,y)\, dx \qquad \text{cont. in all 3 variables}$$

So chain rule applies, and

$$H'(y) = dF \circ G \;\; G' = \left(\int_u^v f_y\, dx, -f, f\right) \circ G \begin{pmatrix} 1 \\ \phi'(y) \\ \psi'(y) \end{pmatrix}$$

$$= \int_{\phi(y)}^{\psi(y)} f_y(x,y)\, dx - f(\phi(y),y)\,\phi'(y) + f(\psi(y),y)\,\psi'(y)$$

$\square$