$$F(y) = \int_a^b f(x,y)\,dx$$

Last time: How to differentiate $F(y)$

Today: Integrate $F(y)$

$$\int_\alpha^\beta F(y)\,dy$$

certainly exists in the sense of Riemann integral because F cont. provided f is.

Q: Does order of integration matter?

Theorem: $f \in C(I)$ $\qquad I = [a,b] \times [\alpha, \beta]$

$$\Rightarrow \int_\alpha^\beta \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_\alpha^\beta f(x,y)\,dy\,dx$$

Proof: Let $\varepsilon > 0$. Since f is unif. cont. on compact set I,

$\exists\, \delta > 0$ s.t. if $d((x,y),(x',y')) < \delta$, then $|f(x,y) - f(x',y')| < \frac{\varepsilon}{(b-a)(\beta-\alpha)}$

Take partition $a = x_0 < x_1 < x_2 \cdots < x_n = b$

$\alpha = y_0 < y_1 < y_2 \cdots < y_m = \beta$

s.t. $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $\operatorname{diam} I_{ij} < \delta$

Define $m_{ij} = \min_{(x,y)\in I_{ij}} f(x,y)$ $\qquad M_{ij} = \max_{I_{ij}} f$

Let $A_{ij} = \text{area}(I_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$

$$m_{ij} A_{ij} \leq \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)\, dy\, dx \leq M_{ij} A_{ij}$$

$$\Rightarrow \sum_{ij} m_{ij} A_{ij} \leq \int_a^b \int_\alpha^\beta f(x,y)\, dy\, dx \leq \sum_{ij} M_{ij} A_{ij}$$

$$\left| \sum m_{ij} A_{ij} - \sum M_{ij} A_{ij} \right| \leq \sum_{ij} A_{ij} \underbrace{\left| m_{ij} - M_{ij} \right|}_{\leq \frac{\varepsilon}{(b-a)(\beta-\alpha)}} = \varepsilon$$

Note: argument above does not depend on the order of integration

$$\Rightarrow \left| \int_a^b \int_\alpha^\beta f(x,y)\, dy\, dx - \int_\alpha^\beta \int_a^b f(x,y)\, dx\, dy \right| < \varepsilon .$$

Since $\varepsilon > 0$ was arbitrary, the integrals must co-incide. $\square$

__Examples:__ 1. $\displaystyle \int_0^1 \underbrace{\int_\alpha^\beta x^y\, dy}\, dx = \int_\alpha^\beta \underbrace{\int_0^1 x^y\, dx}\, dy \qquad 0 < \alpha < \beta$

$$= \int_\alpha^\beta e^{y \ln x}\, dy = \left. \frac{e^{y \ln x}}{\ln x} \right|_\alpha^\beta$$

$$= \frac{x^\beta - x^\alpha}{\ln x}$$

$$= \frac{1}{y+1} x^{y+1} \Big|_{x=0}^{x=1} = \frac{1}{y+1}$$

$$= \ln(y+1) \Big|_\alpha^\beta = \ln \frac{1+\beta}{1+\alpha}$$

$$\Rightarrow \boxed{\int_0^1 \frac{x^\beta - x^\alpha}{\ln x}\, dx = \ln \frac{1+\beta}{1+\alpha}}$$

2. $I = [0,1] \times [\alpha, 1] \qquad \alpha > 0$

$$f(x,y) = \frac{1}{x^2+y^2}$$

$$F(y) = \int_0^1 f(x,y)\,dx = \frac{1}{y} \arctan \frac{1}{y}$$

$$F'(y) = \int_0^1 f_y(x,y)\,dx = \int_0^1 \frac{-2y^2}{(x^2+y^2)^2}\,dx$$

$$= \dots = -\frac{1}{y} \arctan \frac{1}{y} - \frac{1}{y^2+1}$$

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = \dots = -\frac{1}{y^2+1}$$

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy = -\int_0^1 \frac{1}{y^2+1}\,dy = -\frac{\pi}{4}$$

Exchange $x \leftrightarrow y$

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = + \frac{\pi}{4}$$

Order of integration matter !!!

So far, we have integrated over rectangles in $\mathbb{R}^2$

Here: want to look at more general sets, domain

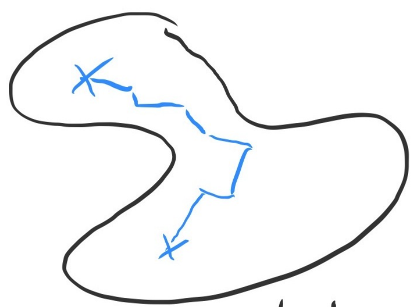
$D \subset \mathbb{R}^n$: open, connected, non-empty

$\downarrow$

Fact: any two points in D can be connected by a polygonal path in D

D

not connected

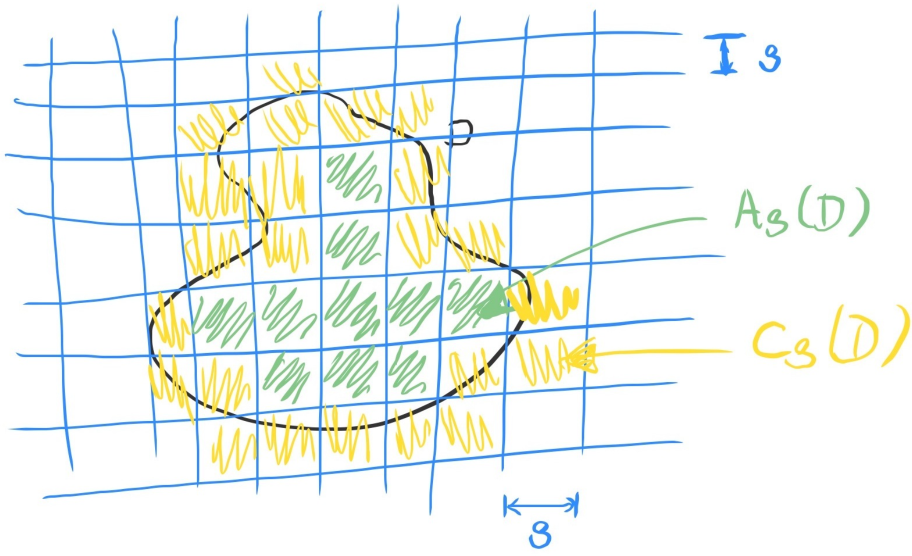


connected

Unit cell $I = [0,1]^n \subset \mathbb{R}^n$

$I_k = k + I$ "unit cell translated by k"

$$\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^k} \delta I_k$$



Want to define "content" (Jordan content, Jordan measure ...)

$$S(I) = 1$$

$$S(p + \delta A) = \delta^n S(A)$$

<u>Def</u>: Given a domain D, we say that D "has content" or "is Jordan measurable" if

$$\lim_{\delta \to 0} S(A_\delta), \quad \lim_{\delta \to 0} S(A_\delta + C_\delta)$$

<u>exist</u> and are equal.

Then $S(D)$ is the common limit, and

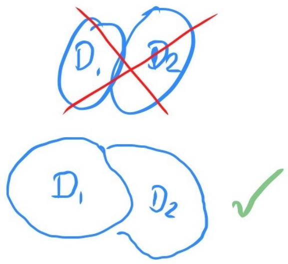
$$\lim_{\delta \to 0} S(C_\delta) = 0 \qquad \text{"}\partial D\text{" has zero content.}$$

$\mathbb{Q}$ what is $\partial\mathbb{Q}$? $\mathbb{R}$

int $\mathbb{Q}$? $\{\}$

We see that a set $D = [0,1] \cap \mathbb{Q}$ is not measurable in the Jordan sense.

Def.: A partition $\mathcal{J} = \{D_j : j = 1, \dots, k\}$ is a collection of subdomains of a domain D if



- $D_j \subset D$ with content
- $\{D_j\}$ disjoint
- $\overline{D} = \bigcup_{j=1}^{k} \overline{D_j}$

Def: $f : \overline{D} \to \mathbb{R}$ bdd.

a Riemann sum for f is

$$\sigma(f, \mathcal{J}; x_1, \dots, x_k) = \sum_{j=1}^{k} f(x_j)\, S(D_j) \quad ; \; x_j \in D_j$$

The "parameter" or "mesh" of $\mathcal{J}$ is $\lambda(\mathcal{J}) = \max \operatorname{diam} D_j$

Def: f is Riemann-integrable on D if $\exists\, I \in \mathbb{R}$ s.t.

$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ s.t. $\forall\, \mathcal{J}$ with $\lambda(\mathcal{J}) < \delta \;\; \forall\, x_j \in D_j$

$$|\sigma(f, \mathcal{J}; x_1, \dots, x_k) - I| < \varepsilon .$$

We write

$$I = \int_D f \, dS$$