Recall: f is Riemann integralle on D $\quad (f \in R(D))$ if $\exists\, I \in \mathbb{R}$ s.t.

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \ \forall J \text{ with } \lambda(J) < \delta \quad \forall x_j \in D_j$$

$$\left| \sum_j f(x_j) \underbrace{S(D_j)}_{\text{in 1D: } \Delta x_j} - I \right| < \varepsilon .$$

Given a partition J, let

$$m_j = \inf_{x \in \overline{D}_j} f(x) \qquad m = \inf_{\overline{D}} f$$

$$M_j = \sup_{\overline{D}_j} f \qquad M = \sup_{\overline{D}} f$$

Riemann's criterion:

$f \in R(D)$ iff $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall J$ with $\lambda(J) < \delta$ $\quad \sum_j (M_j - m_j) S(D_j) < \varepsilon$.

For proofs, please read Kantorovitz etc., not given in detail.

Properties of the Riemann integral

(i) $R(D)$ is a vector space, $\int$ is linear

(ii) $R(D)$ is an $\underbrace{\text{algebra}}_{f,g \in R(D) \Rightarrow fg \in R(D)}$ containing 1, $\quad \int_D 1 \, dS = S(D)$

(iii) $\int$ is monotonic, $m\, S(D) \leq \int_D f\, dS \leq M\, S(D)$

$$\underbrace{\text{monotonic}}_{f \leq g \Rightarrow \int_D f\, dS \leq \int_D g\, dS}$$

$f \leq g \Rightarrow \int_D f\, dS \leq \int_D g\, dS$

(iv) $C(\bar{D}) \subset R(D)$

(v) $f \in C(\bar{D}) \Rightarrow \exists\, p \in \bar{D}$ s.t. $\int_D f\, dS = f(p)\, S(D)$ (MVT)

(vi) $\{D_j\}$ a partition of D, $f \in R(D) \Rightarrow f \in R(D_j)$ and

$$\int_D f\, dS = \sum_j \int_{D_j} f\, dS$$

(vii) $f, g \in R(D)$ $f = g$ on D

$$\Rightarrow \int_D f\, dS = \int_D g\, dS$$

values of f, g on ∂D don't matter !!!

This justifies writing $\int_D f\, dS = \int_{\bar{D}} f\, dS$

(viii) $f \in R(D)$ $g \in C([m, M]) \Rightarrow g \circ f \in R(D)$

(ix) $f \in R(D) \Rightarrow |f| \in R(D)$ and

$$\left| \int_D f\, dS \right| \leq \int_D |f|\, dS$$

Proof of (ii): let $f \in R(D)$

Look at f^2: $\inf_{D_j} f^2 = m_j^2$

Let $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall J$ with $\lambda(J) < \delta$ $\sum_j (M_j - m_j) S_j < \frac{\varepsilon}{M}$

$$\sum_j \underbrace{(M_j^2 - m_j^2)}_{\underbrace{(M_j + m_j)}_{\leq M}(M_j - m_j)} S(D_j) \leq M \underbrace{\sum_j (M_j - m_j) S(D_j)}_{< \frac{\varepsilon}{M}} \leq \varepsilon$$

$\Rightarrow f^2 \in R(D)$

If $f, g \in R(D) \Rightarrow (f-g)^2, (f+g)^2 \in R(D)$, $(f+g)^2 - (f-g)^2 = 4fg \in R(D)$

$\Rightarrow fg \in R(D)$ $\square$

Relationship between R-integral and the iterated integral on $I = [a,b] \times (\alpha, \beta)$

Fact: $f \in R(I)$ and $f(\cdot, y) \in R([a,b])$ for every $y \in [\alpha, \beta]$ then

$$F(y) = \int_a^b f(x,y)\, dx \in R([\alpha, \beta])$$

and

$$\int_I f \, dS = \int_\alpha^\beta \int_a^b f(x,y)\, dx\, dy$$

But: • Existence of iterated integrals does not imply $f \in R(I)$

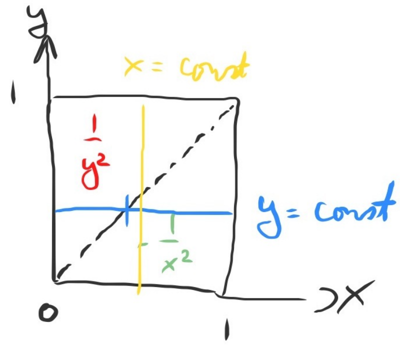
$\boxed{\text{In that case, order of integration may matter!}}$

• $f \in R(I)$ does not imply that iterated integrals exist in R-sense!

(But can be solved by replacing inner integrals by upper or lower R-integrals, or better: use the Lebesgue integral $\to$ Analysis III)

Example for the bad case:

$$f(x,y) = \begin{cases} \frac{1}{y^2} & \text{if } 0<x<y<1 \\ -\frac{1}{x^2} & \text{if } 0<y<x<1 \end{cases}$$



Fix $y>0$:

$$\int_0^1 f(x,y)\,dx = \int_0^y \frac{1}{y^2}\,dx + \int_y^1 \left(-\frac{1}{x^2}\right) dx$$

$$= \frac{1}{y^2}\,y + \frac{1}{x}\Big|_{x=y}^{x=1} = \frac{1}{y} + 1 - \frac{1}{y} = 1$$

Fix $x>0$:

$$\int_0^1 f(x,y)\,dy = \int_0^x \left(-\frac{1}{x^2}\right) dy + \int_x^1 \frac{1}{y^2}\,dy = -\frac{1}{x} - \frac{1}{y}\Big|_{y=x}^{y=1} = -1$$

$$\Rightarrow \int_0^1\int_0^1 f(x,y)\,dx\,dy = 1 \neq \int_0^1\int_0^1 f(x,y)\,dy\,dx = -1$$

$$\Rightarrow f \notin R([0,1]\times[0,1]) \qquad \nabla_0$$

What to do in real life?

Option 1: $f \in C(I) \Rightarrow f \in R(I)$ and iterated integrals exist, so

$$\int_I f \, dS = \int_\alpha^\beta \int_a^b f(x,y) \, dx \, dy = \int_a^b \int_\alpha^\beta f(x,y) \, dy \, dx \qquad (*)$$

Option 2: "Fubini-Tonelli": If "convergent as an improper iterated R-integral"

$$\int_\alpha^\beta \int_a^b |f(x,y)| \, dx \, dy \text{ exists and is finite}, \qquad (**)$$

then $(*)$ holds true and all 3 integrals exist in the sense of Lebesgue. In practice: Check that $(**)$ and iterated integrals exist in the R-sense, then it does not matter what $\int_I f \, dS$ is

For previous example:

$$\int_0^1 |f(x,y)| \, dx = \int_0^y \frac{1}{y^2} \, dx + \int_y^1 \frac{1}{x^2} \, dx = \frac{1}{y} - \frac{1}{x} \Big|_{x=y}^{x=1} = \frac{2}{y} - 1$$

$$\int_0^1 \int_0^1 |f(x,y)| \, dx \, dy = 2 \underbrace{\int_0^1 \frac{1}{y} \, dy}_{\text{this integral does not converge!}} - \int_0^1 dy$$

Fubini-Tonelli criterion fails on this example.