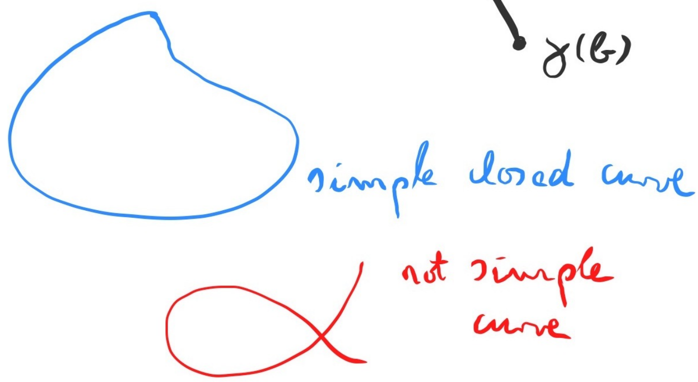
Curves and line integrals

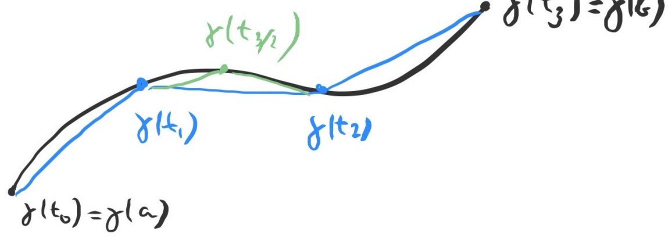
- $\gamma \in C([a,b], \mathbb{R}^k)$ is an (oriented) <u>curve</u> (or path)
- if $\gamma(a) = \gamma(b)$, the curve is <u>closed</u>
- if $\gamma : [a,b)$ is injective, the curve is <u>simple</u>
- curves γ, g are <u>equivalent</u> if $\exists\, h$ s.t. h cont, monotonic, increasing s.t.

  $\gamma = g \circ h$



Partition $\mathcal{J}$: $t_0 = a < t_1 < t_2 \ldots < t_n = b$

$$\lambda(\mathcal{J}) = \max_{i=1,\ldots,n} |\underbrace{t_i - t_{i-1}}_{\Delta t_i}|$$



length relative to partion $\mathcal{J}$: $\Lambda(\mathcal{J}, \gamma) = \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|$

<u>length</u> of curve $\Lambda(\gamma) = \sup_{\mathcal{J}} \Lambda(\mathcal{J}, \gamma)$

If $\Lambda(\gamma) < \infty$, we say that γ is <u>rectifiable</u> ("has length")

<u>Theorem</u>: Let $\gamma \in C^1([a,b], \mathbb{R}^k)$ "smooth curve"

$\Rightarrow \gamma$ rectifiable and $\Lambda(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$

Remark: result extends naturally to piece-wise smooth curves

Proof: $\|\gamma(t_i) - \gamma(t_{i-1})\| = \left\| \int_{t_{i-1}}^{t_i} \gamma'(t)\, dt \right\| \leq \int_{t_{i-1}}^{t_i} \|\gamma'(t)\|\, dt$

now sum from $i = 1, \dots, n$:

$$\Lambda(\mathcal{J}, \gamma) \leq \int_a^b \|\gamma'(t)\|\, dt$$

take sup over all partitions:

$$\Lambda(\gamma) \leq \int_a^b \|\gamma'(t)\|\, dt.$$

To prove the reverse inequality, let $\varepsilon > 0$.

γ', by assumption, is unif. cont. on $[a,b]$: $\exists\, \delta > 0$ s.t. $|s-t| < \delta$

then $\|\gamma'(s) - \gamma'(t)\| < \varepsilon$.

Let $\mathcal{J}$ be a partition with $\lambda(\mathcal{J}) < \delta$.

$$\Rightarrow \|\gamma'(t)\| \leq \|\gamma'(t_i)\| + \varepsilon \quad \text{on } t \in [t_{i-1}, t_i]$$

$$\Rightarrow \int_{t_{i-1}}^{t_i} \|\gamma'(t)\|\, dt \leq \left(\|\gamma'(t_i)\| + \varepsilon\right) \Delta t_i$$

$$= \Big\| \underbrace{\int_{t_{i-1}}^{t_i} \gamma'(t_i)\, dt}_{\int_{t_{i-1}}^{t_i} \gamma'(t)\, dt \; + \; \int_{t_{i-1}}^{t_i} (\gamma'(t_i) - \gamma'(t))\, dt} \Big\| + \varepsilon \Delta t_i$$

$$\leq \Big\| \underbrace{\int_{t_{i-1}}^{t_i} \gamma'(t)\, dt}_{\gamma(t_i) - \gamma(t_{i-1})} \Big\| + 2\varepsilon \Delta t_i$$

Summing over i:

$$\int_a^b \|\gamma'(t)\|\,dt \le \Lambda(J,\gamma) + 2\varepsilon(b-a)$$
$$\le \Lambda(\gamma) + 2\varepsilon(b-a)$$

Since ε was arbitrary: $\displaystyle\int_a^b \|\gamma'(t)\|\,dt \le \Lambda(\gamma)$ $\square$

<u>Note:</u> $\Lambda(\gamma)$ is independent of the parameterization. Indeed, when

$\gamma = g\circ h$, $h \in C^1$, strictly increasing:

$$\int_a^b \|\gamma'(t)\|\,dt = \int_a^b \|\underbrace{\tfrac{d}{dt}(g\circ h)}_{g'\circ h\, h'}\|\,dt = \int_a^b \|g'\circ h\|\, h'(t)\,dt = \int_{h(a)}^{h(b)} \|g'(u)\|\,du$$

$u = h(t) \Rightarrow du = h'(t)\,dt$

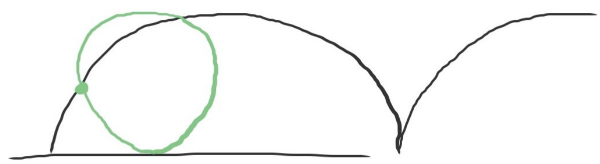
<u>Examples:</u> ① Circumference of a circle:

$\gamma(t) = R(\cos t, \sin t)$ $\quad t \in [0, 2\pi]$

$\gamma'(t) = R(-\sin t, \cos t) \Rightarrow \|\gamma'(t)\| = R$

$$\Rightarrow \Lambda(\gamma) = \int_0^{2\pi} R\,dt = 2\pi R$$

② cycloid

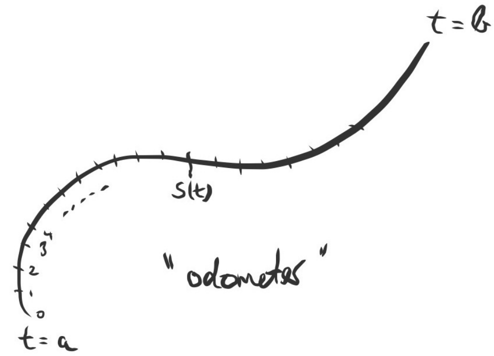


$\gamma(t) = (t - \sin t, 1 - \cos t)$

$\gamma'(t) = (1 - \cos t, \sin t)$

$$\Rightarrow \|\gamma'(t)\|^2 = (1-\cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t$$
$$= 2(1 - \cos t) = 4\sin^2 \tfrac{t}{2}$$

$$\Rightarrow \Delta(\gamma) = 2\int_0^{2\pi} \sin\frac{t}{2}\, dt = -4\cos\frac{t}{2}\Big|_0^{2\pi} = 8$$



Arc-length parameterization

$$s(t) = \int_a^t \|\gamma'(\tau)\|\, d\tau$$

$s'(t) = \|\gamma'(t)\| > 0$ for non-degenerate parameterization

by monotonicity $s(t)$ is invertible.

Let $\gamma(t(s))$ be the arclength parameterization

Line integral of $f \in C(\gamma, \mathbb{R})$

$$\boxed{\begin{aligned}\int_\gamma f\, ds &:= \int_0^{\Delta(\gamma)} f(t(s))\, ds \\ &= \int_a^b f(t)\, \|\gamma'(t)\|\, dt\end{aligned}}$$

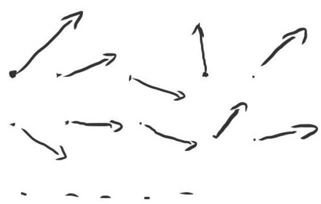
$u = t(s)$

$\frac{du}{ds} = t'(s) = \frac{1}{s'(t)} = \frac{1}{\|\gamma'(t)\|}$

$ds = \|\gamma'(u)\|\, du$

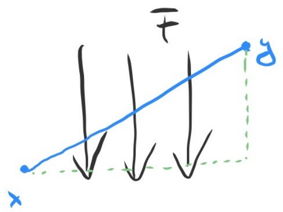
Note: $\int_\gamma f\, ds$ is independent of the parameterization of γ. (HW)

Vector fields: $F : D \subset \mathbb{R}^k \to \mathbb{R}^k$



E.g. Force field

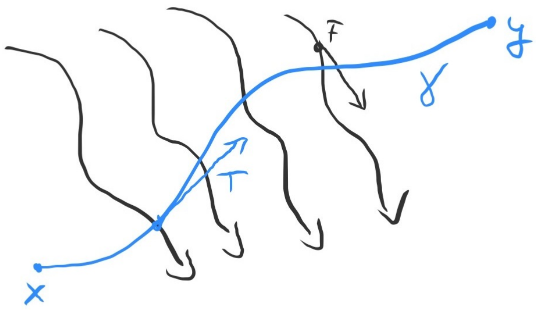
Work done on a piece of mass m, when moving from point x to y in a <u>constant</u> force field F



work done in this procedure is $m \cdot$ displacement <u>in the direction of the force</u>.

$$W = m(x-y)\cdot F \quad \text{or} \quad m\langle x-y, F\rangle$$

For non-constant F



$$W = \int_\gamma \langle F, T\rangle \, ds =: \int_\gamma F \cdot dx$$

T is a unit tangent vector

$$\int_\gamma \langle F, T\rangle \, ds \qquad T = \frac{\gamma'}{\|\gamma'\|}$$

$$= \int_\gamma \left\langle F, \frac{\gamma'}{\|\gamma'\|}\right\rangle ds$$

$$= \int_a^b \left\langle F\circ\gamma, \cancel{\frac{\gamma'}{\|\gamma'\|}}\right\rangle \cancel{\|\gamma'(t)\|}\, dt$$

$$= \int_a^b \langle F\circ\gamma, \gamma'\rangle \, dt = \boxed{\int_a^b F(\gamma(t))\cdot\gamma'(t)\, dt}$$

line integral for vector fields.