Most relevant topics:

- Look at mock midterm questions
  - power series, radius of convergence
  - tricks: geometric series
  - Taylor (1 variable, several variables)
  - CHAIN RULE !
  - inverse & implicit function theorem
- Uniform convergence / uniform continuity
  - → Exchange of limit and integration
  - → Exchange of differentiation and integration

- Integration over domains
  - → change of variable formula, polar coordinates
- Line integrals → Green's Theorem

To come:

- Surface integrals → The divergence Theorem
  Stokes' Theorem

Applications ....

Recall: $D \subset \mathbb{R}^k$ a domain with content, γ a curve

- Line integral of a function $f \in C(D, \underline{\underline{\mathbb{R}}})$

$$\int_\gamma f \, ds = \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt$$

Special case $f = 1$: Arc-length $\Lambda(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$

- Line integral for a vector field $F \in C(D, \underline{\underline{\mathbb{R}^k}})$

$$\int_\gamma F \cdot dx = \int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt$$

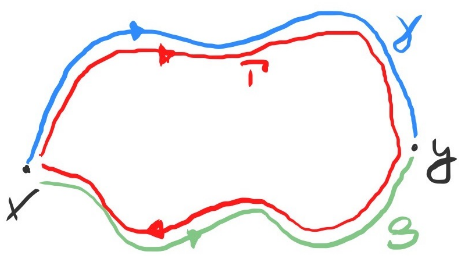
project F onto tangent of γ, then use first definition

<u>Def</u>: $F \in C(D, \mathbb{R}^k)$

F conservative if $\int_\gamma F \cdot dx$ depends only on the end-points $\gamma(a)$ and $\gamma(b)$; γ piece-wise smooth.

<u>Remark</u>: This is equivalent to saying

F conservative iff $\int_\gamma F \cdot dx = 0$ for γ closed, piecewise smooth



$$\int_\gamma F \cdot dx = \int_S F \cdot dx$$

$$\int_\Gamma F \cdot dx = \int_\gamma F \cdot dx - \int_S F \cdot dx$$
$$= 0$$

Theorem: $F \in C(D, \mathbb{R}^k)$ conservative iff $\exists\, \phi \in C^1(D, \mathbb{R})$ s.t.

$$F = \nabla \phi$$

potential

Proof: "$\Leftarrow$":
$$\int_\gamma F \cdot dx = \int_a^b \underbrace{F(\gamma(t)) \cdot \gamma'(t)}\, dt$$

$$= (\nabla\phi)(\gamma(t)) \cdot \gamma'(t)$$

$$= \frac{d}{dt}\left(\phi(\gamma(t)) \right)$$

$$\overset{\text{F.T.C.}}{=} F(\gamma(b)) - F(\gamma(a))$$

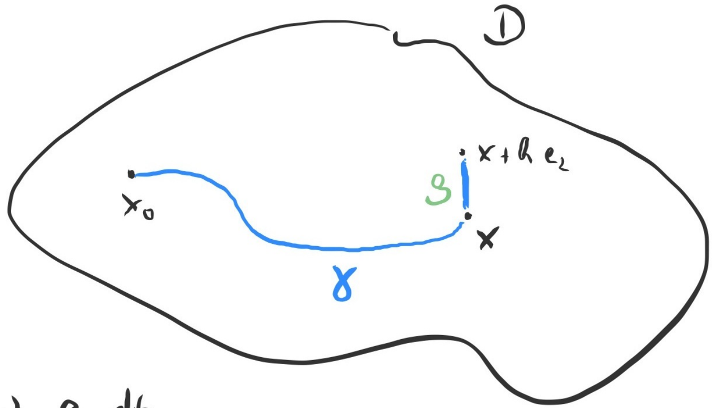
So line integral depends only on $\gamma(b)$ and $\gamma(a)$.

"$\Rightarrow$": Suppose F is conservative, fix $x_0 \in D$

Define:

$$\phi(x) = \int_\gamma F \cdot dx$$ where γ is any curve, smooth, connecting x_0 and x.

$$\phi(x + h e_i) = \int_{\gamma \cup S} F \cdot dx$$



$$= \phi(x) + \underbrace{\int_S F \cdot dx}_{\int_0^h F(\underbrace{x + t e_i}_{S(t)}) \cdot \underbrace{e_i}_{= S'(t)}\, dt}$$

$$\phi(x+he_i) = \phi(x) + \int_0^h F_i(x+te_i)\, dt$$

$$\frac{\partial \phi}{\partial x_i} = \frac{d}{dh}\phi(x+he_i)\Big|_{h=0} = F_i(x+he_i)\Big|_{h=0} = F_i(x)$$

$\Rightarrow \quad \nabla \phi = F$

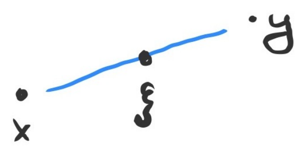
also: $\phi \in C^1(D, \mathbb{R})$ $\square$

Remarks: ① $\alpha = F \cdot dx$ is a differential 1-form

α is "exact" $\Leftrightarrow$ F is "conservative"

② If $F = \nabla \phi = \nabla \psi$,

then $\phi - \psi = \text{const}$ on D



Note: $\Theta(x) - \Theta(y) = \nabla \Theta(\xi) \cdot (x-y)$ for some ξ on the line segment from x to y

(MVT)

Recall: D is connected $\Rightarrow$ every two points can be connected by a polygonal path.

Here: $\Theta = \phi - \psi \quad \Rightarrow \quad \nabla \Theta = 0$

$\Rightarrow$ along a straight line segment, $\Theta = \text{const}$

$\Rightarrow \Theta = \text{const}$ on D

Cor. $F \in C^1(D, \mathbb{R}^k)$ conservative $\Rightarrow$ DF symmetric

Proof: $F = \nabla\phi \Rightarrow DF = D\nabla\phi = \text{Hess}\,\phi$ which is symmetric $\square$

Note: DF symmetric is not sufficient for F conservative: E.g.:

$$F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

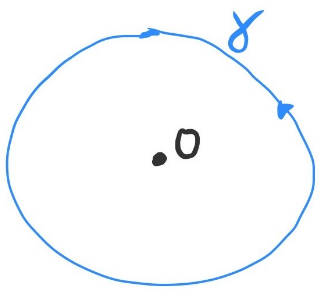
$$\left.\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{-1\,(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial F_2}{\partial x} &= \frac{1\cdot(x^2+y^2) - 2x\cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}\right\} \text{ on } D = \mathbb{R}^2 \setminus \{0\}$$

$\Rightarrow DF$ is symmetric

Take γ to be the unit circle

$\gamma = (\cos t, \sin t)$ $\quad t \in [0, 2\pi]$ $\quad \gamma' = (-\sin t, \cos t)$

$$\int_\gamma F \cdot dx = \int_0^{2\pi} \underbrace{(-\sin t \ \ \cos t) \cdot \gamma'(t)}_{=1}\, dt = 2\pi \neq 0 \ \overset{\nabla}{\circ}$$

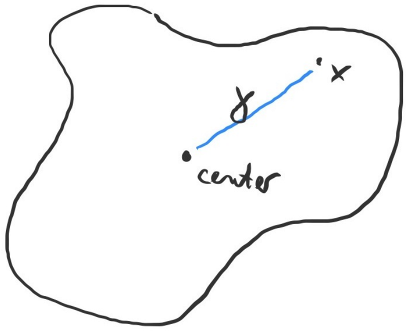


"not simply connected"

closed curves can not be cont. contracted to a point.

Theorem: D star-shaped

Then $F \in C^1(D, \mathbb{R}^k)$ conservative $\Leftrightarrow$ DF symmetric



Proof. WLOG, let the center of D be 0.

$$\phi(x) = \int_\gamma F \cdot dx$$

$\nwarrow$ here: γ is the straight line segment

every $x \in D$ can be connected to center by a straight line segment

$$= \int_0^1 F(\underbrace{tx}_{\gamma}) \cdot \underbrace{x}_{\gamma'} \; dt$$

$$\frac{\partial \phi}{\partial x_i} = \int_0^1 \left(\frac{\partial F}{\partial x_i}(tx)\; t \cdot x + F(tx) \cdot e_i \right) dt$$

Recall integrand:

$$\underbrace{\frac{\partial F}{\partial x_i}(tx) \cdot x \; t}\; + F_i(tx)$$

$$= \sum_j x_j \underbrace{\frac{\partial F_j}{\partial x_i}(tx)}_{= \frac{\partial F_i}{\partial x_j}(tx) \text{ by assumption}}$$

$$= \frac{d}{dt}\left(t\, F_i(tx)\right)$$

$$\Rightarrow \frac{\partial \phi}{\partial x_i} = \int_0^1 \frac{d}{dt}\left(t\, F_i(tx)\right) dt = t F_i(tx) \Big|_{t=0}^{t=1} = F_i(x)$$

$$\Rightarrow \nabla \phi = F \quad \Longrightarrow \quad F \text{ is conservative!} \qquad \square$$