Recall: Smooth surface M parameterized by $f \in C(\bar{U}, \mathbb{R}^3) \cap C^1(\bar{U}, \mathbb{R}^3)$

Normal field: $n = \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$ $\quad f = f(u,v) \quad (u,v) \in \bar{U}$

$\hat{n} = \frac{n}{\|n\|}$ (well-defined for smooth M)

Surface integrals

area $\sigma(M) = \int_{\bar{U}} \|n\| \, dS$

surface integral: $\phi \in C(M, \mathbb{R})$

$$\int_M \phi \, d\sigma = \int_{\bar{U}} \phi \circ f \, \|n\| \, dS$$

$F \in C(M, \mathbb{R}^3)$: flux integral

$$\int_M F \cdot \hat{n} \, d\sigma = \int_{\bar{U}} F \circ f \cdot n \, dS$$

sign depends on choice of parametrization!

Line integrals

length $\Lambda(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$

line integral $\int_\gamma f \, ds = \int_a^b f \circ \gamma \, \|\gamma'(t)\| \, dt$

$$\int_\gamma F \cdot dx = \int_\gamma F \cdot \hat{t} \, ds = \int_a^b F \circ \gamma \cdot \gamma'(t) \, dt$$

Examples: 1. M upper hemisphere of radius 1, outward normal field, centered at 0

$\phi(x,y,z) = (x^2+y^2) z$

Use parameterization $f(\theta, \phi) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$

$\theta \in [0, \frac{\pi}{2}]$, $\phi \in [0, 2\pi]$

Recall: $\|n\| = \sin\theta$

$$\int_M \phi \, d\sigma = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^2\theta \, \underbrace{(\cos^2\phi + \sin^2\phi)}_{1} \cos\theta \, \underbrace{\sin\theta}_{\|n\|} \, d\theta \, d\phi$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin^3\theta \cos\theta \, d\theta$$

$u = \sin\theta \Rightarrow du = \cos\theta \, d\theta$

$$= 2\pi \underbrace{\int_0^1 u^3 \, du}_{\frac{1}{4}} = \frac{\pi}{2}$$

2. Same M,

$$F = \frac{1}{x^2+y^2+z^2}(1,1,1)$$

Here: $F \circ f = (1,1,1)$

Recall: $$n = \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi} = \begin{pmatrix} \sin^2\theta \cos\phi \\ \sin^2\theta \sin\phi \\ \sin\theta \cos\theta \end{pmatrix}$$

$$\Rightarrow \int_M F \cdot \hat{n} \, d\sigma = \int_{\bar{U}} F\circ f \cdot n \, dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Big(\underbrace{\cancel{\sin^2\theta \cos\phi}}_{=0} + \underbrace{\cancel{\sin^2\theta \sin\phi}}_{=0 \text{ (integral over full period of } \sin\phi \text{ resp. } \cos\phi)} + \sin\theta\cos\theta \Big) \, d\theta \, d\phi$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta = 2\pi \underbrace{\int_0^1 u \, du}_{=\frac{1}{2}} = \pi$$

<u>Divergence Theorem</u>: $D \subset \mathbb{R}^k$ domain, $V \subset D$ domain s.t.

$\bar{V} \subset D$, bounded, regular

$\partial \bar{V}$ has non-vanishing piece-wise cont. normal field n

$F: C^1(\bar{V}, \mathbb{R}^k)$

$$\int_{\partial \bar{V}} F \cdot \hat{n} \, d\sigma = \int_V \nabla \cdot F \, dx$$

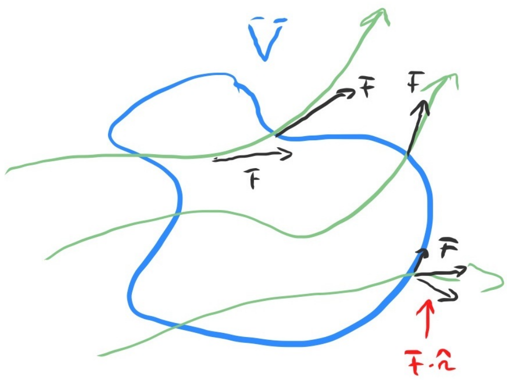
$\hat{n}$: <u>outward</u> unit normal vector

Application: Derivation of the diffusion equation

Let $u : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a concentration (mass/volume)

$$m = \int_{\hat V} u \, dx \quad , \quad \hat V \subset D \text{ subdomain, arbitrary}$$

"total mass contained in $\hat V$"



Rate of change of mass:

$$\frac{dm}{dt} = -\int_{\partial \hat V} F \cdot \hat n \, d\sigma$$

"changes in total mass are due to flux through the boundary"

For diffusion: "Fick's law" $F = -\nabla u$

$$\Rightarrow \int_{\hat V} \frac{\partial u}{\partial t} \, dx = \int_{\partial \hat V} \hat n \cdot \nabla u \, d\sigma = \int_{\hat V} \underbrace{\nabla \cdot \nabla u}_{= \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}} dx \Longrightarrow \boxed{\frac{\partial u}{\partial t} = \Delta u}$$

"diffusion equation" or "heat equation"

Stokes' theorem: $D \subset \mathbb{R}^3$

$M \subset D$ smooth surface ($\Rightarrow$ non-vanishing normal field), bounded, orientable

∂M has smooth parameterization, and orientation anti-clockwise w.r.t. normal field

$F \in C^1(\bar D, \mathbb{R}^3)$

$$\Rightarrow \int_{\partial M} F \cdot dx = \int_M (\nabla \times F) \cdot \hat n \, d\sigma$$

Compare: Green's Theorem: $D \subset \mathbb{R}^2$

$$\int_{\partial D} F \cdot dx = \int_D \nabla^{\perp} \cdot F \, dS$$

Green's Theorem is Stokes' theorem with $F = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}$

$n = e_3$

Proof: WLOG, let $F = \begin{pmatrix} 0 \\ 0 \\ F_3 \end{pmatrix}$. $f(u,v)$ parameterizes M $\quad f : \bar{U} \to M$

$\gamma(t)$ " $\partial \bar{U}$ $\quad \gamma : [a,b] \to \partial \bar{U}$

$\underbrace{f \circ \gamma}_{g} : [a,b] \to \partial M$

$g' = Df \circ \gamma \cdot \gamma'$

$$\int_{\partial M} F \cdot dx = \int_a^b F \circ g \cdot g' \, dt$$

$$= \int_a^b \underbrace{F \circ f \circ \gamma \cdot Df \circ \gamma}_{\phi \circ \gamma} \, \gamma' \, dt$$

$$= \int_{\partial \bar{U}} \phi \cdot du \overset{\text{green's}}{=} \int_{\bar{U}} \nabla^{\perp} \cdot \phi \, dS$$

$$= \int_{\bar{U}} \underbrace{\nabla^{\perp} F_3 \circ f \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}}_{(\nabla \times F) \circ f \cdot n \, dS} dS$$

$$= \int_M (\nabla \times F) \cdot \hat{n} \, d\sigma$$

$\square$

$D = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right)$

$$\nabla^{\perp} \cdot \phi = \nabla^{\perp} \cdot (F \circ f \; Df)$$

here $F = \begin{pmatrix} 0 \\ 0 \\ F_3 \end{pmatrix}$

$$= \nabla^{\perp} \cdot (F_3 \circ f \; Df_3)$$

$$= \nabla^{\perp}(F_3 \circ f) \cdot Df_3 + F_3 \circ f \; \underbrace{\nabla^{\perp} \cdot Df_3}_{=0}$$

(check in coordinates, or by curl grad $= 0$)

$$= DF_3 \circ f \; \nabla^{\perp} f \cdot Df_3$$

$$= (\partial_1, \partial_2) F_3 \circ f \; \nabla^{\perp} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \cdot Df_3$$

$$= \nabla^{\perp} F_3 \circ f \; \underbrace{\nabla^{\perp} \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \cdot Df_3}_{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}}$$

$\nabla^{\perp} f_3 \cdot Df_3 = 0$ by orthogonality