Divergence theorem:

$$\int_V \nabla \cdot F \, dx = \int_{\partial V} \hat{n} \cdot F \, d\sigma$$

outward unit normal on ∂V

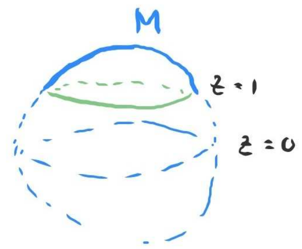
Stokes' theorem

$$\int_S (\nabla \times F) \cdot \hat{n} \, d\sigma = \int_{\partial S} F \cdot dx$$

anticlockwise w.r.t. $\hat{n}$

"right-hand rule"

Example: $F(x) = (y^2, z^2, x^2)$



$M = \{(x,y,z) : x^2 + y^2 + z^2 = 4, \ z \geq 1\}$

$\partial M = \{(x,y,1) : x^2 + y^2 = 3\}$

parameterize: $\gamma(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t, 1)$

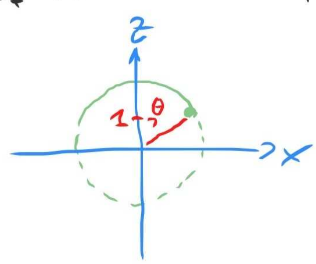
$$\gamma'(t) = (-\sqrt{3}\sin t, \sqrt{3}\cos t, 0)$$

$$\Rightarrow \int_\gamma F \cdot dx = \int_0^{2\pi} \underbrace{(3\sin^2 t, 1, 3\cos^2 t)}_{F \circ \gamma} \cdot \underbrace{(-\sqrt{3}\sin t, \sqrt{3}\cos t, 0)}_{\gamma'(t)} \, dt$$

$$= \sqrt{3} \int_0^{2\pi} (3\sin^3 t + \cos t) \, dt = 0$$

by integrating over full periods of sin and cos functions

Compute flux integral:



$$\nabla \times F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} = -2 \begin{pmatrix} z \\ x \\ y \end{pmatrix}$$

Spherical coordinates for M: $f(\theta, \phi) = 2(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\phi \in [0, 2\pi]$

$\theta \in [0, \frac{\pi}{3}]$

Limit value for θ: $1 = \underbrace{2\cos\theta}_{= z} \Rightarrow \theta = \frac{\pi}{3}$

Recall:

$$n = \frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi} = 4 \begin{pmatrix} \sin^2\theta \cos\phi \\ \sin^2\theta \sin\phi \\ \cos\theta \sin\theta \end{pmatrix}$$

$$\int_M (\nabla \times F) \cdot \hat{n} \, d\sigma = \int_{\bar{U}} (\nabla \times F) \circ f \cdot n \, du \qquad u = \begin{pmatrix} \theta \\ \phi \end{pmatrix} \qquad \bar{U} = [0, \tfrac{\pi}{3}] \times [0, 2\pi]$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} -2 \, (\underbrace{\cos\theta}_{z}, \underbrace{\sin\theta\cos\phi}_{x}, \underbrace{\sin\theta\sin\phi}_{y}) \cdot 4 \, (\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \cos\theta\sin\theta) \, d\theta \, d\phi$$

$= 0$ (look at ϕ-integrals first – all are over a full period of sin or cos.)

<u>Cor.</u> Suppose M (as in Stokes' theorem) has no boundary.

$$\Rightarrow \int_M (\nabla \times F) \cdot \hat{n} \, d\sigma = 0$$

for <u>any</u> smooth vector field F.

<u>Cor.</u> F a smooth vector field on a simply connected domain $D \subset \mathbb{R}^3$

F is conservative $\Leftrightarrow$ $\nabla \times F = 0$

<u>Proof</u>: "$\Rightarrow$": $F = \nabla\phi \Rightarrow \nabla \times F = \nabla \times \nabla\phi = 0$

HW: Show that $\nabla \times \nabla = 0$ in general

"$\Leftarrow$": Let γ be a closed curve

M be a "capping surface" s.t. $\partial M = \gamma$

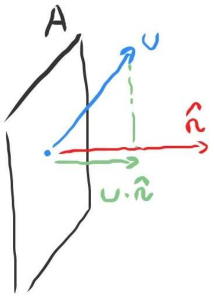
e.g. "soap film", existence under stated assumptions will be shown in a topology class.

$$\Rightarrow \int_\gamma F \cdot dx = \int_S \underbrace{(\nabla \times F)}_{=0} \cdot \hat{n} \, d\sigma = 0 \Rightarrow F \text{ conservative.}$$

$\square$

Interpretation of divergence and curl.

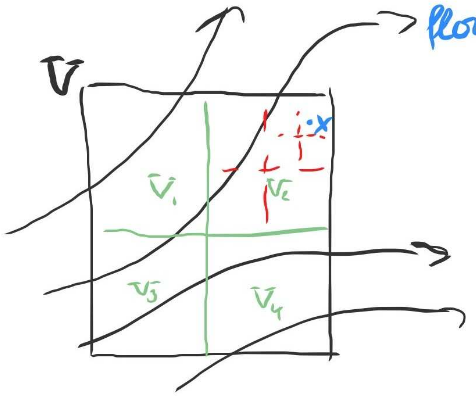
Take vector field u which denotes velocity, const for now



$$\underbrace{u\cdot\hat{n}}\, A = \frac{\text{volume}}{\text{time}}$$ passing through surface A (volume flux)

$\frac{\text{volume}}{\text{area}\cdot\text{time}}$: volume flux (area) density (volume passing through surface per area per time)

Let $\bar{V}$ is a domain, then $\int_{\partial\bar{V}} u\cdot\hat{n}\, d\sigma$ is the volume flux through $\partial\bar{V}$



flow lines (u is tangent to these curves)

source density of volume $\quad \operatorname{div} u(x) = \lim_{r\to 0} \dfrac{\int_{\partial B(x,r)} u\cdot\hat{n}\, d\sigma}{\operatorname{Vol} B(x,r)}$

Divergence Theorem: $\quad \operatorname{div} u(x) = \lim_{r\to 0} \dfrac{\int_{B(x,r)} \nabla\cdot u\, dx}{\operatorname{Vol} B(x,r)} = \lim_{r\to 0} \operatorname{Av}_{B(x,r)} \nabla\cdot u = \nabla\cdot u(x)$

Conclusion: $\operatorname{div} u = \nabla\cdot u$ is the source density of volume.

In particular: a flow is contracting if $\operatorname{div} u < 0$, expanding if $\nabla\cdot u > 0$.

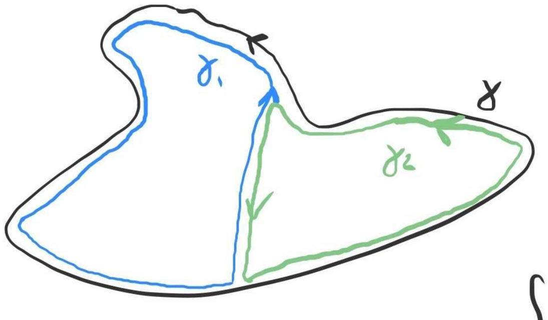
Circulation:

γ simple closed curve, u "velocity field".

$$C = \int_\gamma u \cdot dx$$

"net spinning of flow around curve"

Remark: conservative vector field "don't spin".



$$C(\gamma) = C(\gamma_1) + C(\gamma_2)$$

"spinning (area-) density" $\hat{n} \cdot \operatorname{curl} u(x) = \lim_{r \to 0} \frac{\int_{\partial D(x,r)} u \cdot dx}{\sigma(D(x,r))} \overset{\text{Stokes'}}{=} \lim \frac{\int_{D(x,r)} (\nabla \times u) \cdot \hat{n} \, d\sigma}{\sigma(D(x,r))} = (\nabla \times u) \cdot \hat{n}$

lies in plane with unit normal $\hat{n}$