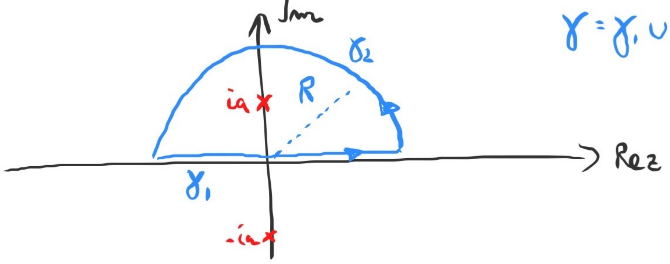
① Typical application of the residue theorem: Compute

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^4} \qquad a \in \mathbb{R},\ a \neq 0 \qquad f(z) = \frac{1}{(z^2+a^2)^4}$$



$\gamma = \gamma_1 \cup \gamma_2$

To parametrize γ_2, use complex polars: $z = Re^{i\theta}$ $\quad \theta = 0 \dots \pi$

$dz = iRe^{i\theta}\, d\theta$

First observation: $$\left| \int_{\gamma_2} f(z)\, dz \right| = \left| \int_0^\pi \frac{1}{(R^2e^{2i\theta}+a^2)^4}\, iRe^{i\theta}\, d\theta \right|$$

$$\leq \int_0^\pi \frac{1}{|R^2e^{2i\theta}+a^2|^4}\, R\, d\theta$$

$$\leq \frac{R}{|R^2-a^2|^4} \int_0^\pi d\theta \xrightarrow[R\to 0]{} 0$$

To use residue theorem, need $\mathrm{Res}(f, ia)$, so write out Laurent series of f centered at ia: $\quad z = ia + \zeta$

$$f(z) = \frac{1}{((ai+\zeta)^2+a^2)^4} = \frac{1}{(-\cancel{a^2}+2ai\zeta+\zeta^2+\cancel{a^2})^4} = \frac{1}{\zeta^4}\frac{1}{(2ai+\zeta)^4}$$

$$= \frac{1}{(2ai\zeta)^4} \underbrace{\left(1+\frac{\zeta}{2ai}\right)^{-4}}_{= \sum_{k=0}^{\infty} \binom{-4}{k} \left(\frac{\zeta}{2ai}\right)^k}$$

$$\boxed{(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k}$$

Need to pick coefficient corresponding to ζ^{-1}, i.e. $k=3$

$$\Rightarrow \mathrm{Res}(f, ia) = \frac{1}{(2ai)^4}\binom{-4}{3}\frac{1}{(2ai)^3} = \frac{i}{(2a)^7}\,\frac{(-4)(-4-1)(-4-2)}{1\cdot 2\cdot 3} = \dots = \frac{-i5}{32a^7}$$

$$\int_{\gamma} f(z)\,dz = 2\pi i \,\mathrm{Res}(f, ai) = \frac{5\pi}{16a^7}$$

$$\|$$

$$\int_{-R}^{R} \frac{dx}{(x^2+a^2)^4} + \underbrace{\int_{\gamma_2} f(z)\,dz}_{\to 0 \text{ as } R\to\infty}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^4} = \frac{5\pi}{16a^7}$$

(2) Boundary condition for Poisson's equation:

$$-\Delta u = f \qquad \text{in } D \subset \mathbb{R}^2 \quad \text{bounded domain}$$

Q: what boundary conditions make this (essentially) unique?

Computation for $u \in C^2(\bar{D})$

Suppose that u_1, u_2 are solutions. Then $-\Delta(u_1 - u_2) = 0$ $\qquad \Delta = \nabla\cdot\nabla$

$$\int_D (u_1-u_2)(-\underbrace{\Delta(u_1-u_2)}_{=0})\,dx = -\int_{\partial D} (u_1-u_2)\, n\cdot\nabla(u_1-u_2)\,dS + \int_D \underbrace{\nabla(u_1-u_2)\cdot\nabla(u_1-u_2)}_{\|\nabla(u_1-u_2)\|^2}\,dx$$

outward unit normal

Suppose boundary term vanishes:

$$0 = \int_D \|\nabla(u_1-u_2)\|^2\,dx \Rightarrow \|\nabla(u_1-u_2)\| = 0 \Rightarrow \nabla(u_1-u_2) = 0$$

$$\Rightarrow u_1 = u_2 + \text{const}$$

Sufficient conditions for the boundary integral to vanish are two natural alternative boundary conditions:

(a) $u = g$ on ∂D , g a given function "Dirichlet problem"

(b) $n \cdot \nabla u = g$ on ∂D , " " " " "Neumann problem"

Existence of solutions under these boundary conditions can be shown

→ PDE class

③ Helmholtz - decomposition of vector fields:

Let w be a C^1 vector field on $D \subset \mathbb{R}^n$, bounded. Then there exist a unique vector field u and a function ϕ s.t.

$$\underbrace{w = u}_{\text{divergence-free}} + \underbrace{\nabla\phi}_{\text{curl-free}} \qquad \text{where} \qquad \nabla \cdot u = 0 \quad \text{in } D, \qquad u \cdot n = 0 \quad \text{on } \partial D$$

Remark: $\nabla \times w = \nabla \times u + \underbrace{\nabla \times \nabla\phi}_{=0}$

Remark: $u \perp \nabla\phi$ in the L^2-sense:

$$\int_D u \cdot \nabla\phi \, dx = \int_{\partial D} \underbrace{n \cdot u}_{=0} \phi \, dS - \int_D \underbrace{\nabla \cdot u}_{=0} \phi \, dx = 0$$

Proof: (a) Uniqueness: suppose

$$w = u_1 + \nabla\phi_1 = u_2 + \nabla\phi_2$$

$$\Rightarrow (u_1 - u_2) + \nabla(\phi_1 - \phi_2) = 0$$

By orthogonality, $u_1 - u_2 = 0$ and $\underbrace{\nabla(\phi_1 - \phi_2) = 0}_{\Rightarrow \phi_1 = \phi_2 + \text{const}}$

(b) Existence: Take divergence of $w = u + \nabla\phi$

$$\nabla\cdot w = \underbrace{\nabla\cdot u}_{=0} + \Delta\phi$$

$$\Rightarrow \left.\begin{array}{lll} \Delta\phi = \nabla\cdot w & \text{in } D & (*) \\ n\cdot\nabla\phi = n\cdot w & \text{on } \partial D & (**) \end{array}\right\}$$

Neumann problem, existence from PDE theory

$n\cdot u = n\cdot w - n\cdot\nabla\phi$ from (**)

$\nabla\cdot u = \nabla\cdot w - \Delta\phi = 0$ from (*)

$\uparrow$

$u := w - \nabla\phi$

$\uparrow$

(4) Suppose $f \not\equiv 0$

$$-\Delta f = \lambda f \qquad \text{with } f = 0 \text{ on } \partial D$$

"eigenfunction of the Dirichlet - Laplacian"

Then

$$\underbrace{\int_D \underbrace{\|\nabla f\|^2}_{\nabla f\cdot\nabla f} dx}_{>0} = \underbrace{\int_{\partial D} \overset{=}{\underset{0}{f}}\, n\cdot\nabla f \, dS}_{=0} - \int_D f \underbrace{\Delta f}_{-\lambda f} dx = \lambda \underbrace{\int_D f^2 dx}_{>0}$$

$\Rightarrow \lambda > 0$ $\Rightarrow$ all eigenvalues of the D-Laplacian are positive

"$-\Delta$" is a positive operator