Advanced Calculus and Methods of Mathematical Physics

Homework 10

Not for credit

Note: This homework comprises miscellaneous subject areas from the last two weeks of classes. This homework will not count toward bonus credit, but it's very much worth working on. I will publish sample solutions on Monday, May 18.

1. Suppose $f, g\colon [0, 2\pi] \to \mathbb{C}$ are Riemann-integrable. Prove the following properties of the Fourier transform.

   (a) If f is real-valued, then $f_{-k} = \overline{f_k}$.

   (b) If f is continuously differentiable, then $(f')_k = \mathrm{i}k\, f_k$.

   (c) Let $T_a f$ denote the right-translation by $a \in \mathbb{R}$ of f, i.e., $(T_a f)(x) = f(x - a)$, with the understanding that f is periodically extended outside of its fundamental domain $[0, 2\pi]$. Show that
   $$(T_a f)_k = \mathrm{e}^{-\mathrm{i}ka}\, f_k\,.$$

   (d) Let
   $$(f * g)(x) = \int_0^{2\pi} f(y)\, g(x - y)\, \mathrm{d}y$$
   denote the *convolution* of the functions f and g, again with the understanding that the functions are periodically extended outside of their fundamental domain. Show that
   $$(f * g)_k = 2\pi\, f_k\, g_k\,.$$

2. Compute the Fourier transform of the "saw-tooth function" $f(x) = x$ on $[-\pi, \pi)$, periodically extended outside its fundamental domain.

3. Which of the following functions are complex-differentiable?

   (a) $f(z) = z^2$,

   (b) $f(z) = |z|^2$,

   (c) $f(z) = \cos(z)$.

4. Show that if $f(z) = u(x, y) + \mathrm{i}\, v(x, y)$ with $z = x + \mathrm{i}y$ is complex-differentiable (holomorphic) on some domain $D \subset \mathbb{C}$, then u and v are harmonic functions on $\mathbb{R}^2$, i.e., $\Delta u = \Delta v = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator on the corresponding domain of $\mathbb{R}^2$.

   Remark: That adds another item to the list of fundamental equivalences in complex variable calculus: "harmonic" $\iff$ "complex-differentiable (holomorphic)" $\iff$ "convergent Taylor series (analytic)" $\iff$ "path independence of complex line integral (identification with conservative vector field)".

5. Use the residue theorem to compute

$$\int_{|z|=1} z^2 \sin \frac{1}{z} \, \mathrm{d}z \, .$$

6. Let D be a domain in $\mathbb{R}^n$. A function $f \in C^2(D)$ is called *harmonic* if $\Delta f = 0$, where Δ is the *Laplace operator* defined via

$$\Delta f = \nabla \cdot \nabla f = \partial_1^2 f + \cdots + \partial_n^2 f \, .$$

   Show that a harmonic function has the mean value property

$$f(x) = \frac{1}{S(\partial B(x,r))} \int_{\partial B(x,r)} f \, \mathrm{d}S$$

   for every $x \in D$ and every $r > 0$ sufficiently small such that $\partial B(x, r) \subset D$, where $B(x, r)$ denotes the ball centered at x with radius r and

$$S(\partial B(x,r)) = \int_{\partial B(x,r)} \mathrm{d}S$$

   is the $(n-1)$-dimensional content of $\partial B(x, r)$.

   Hint: Proceed in the following steps:

   (a) Define

$$\phi(r) = \frac{\displaystyle\int_{\partial B(x,r)} f \, \mathrm{d}S}{\displaystyle\int_{\partial B(x,r)} \mathrm{d}S} \, .$$

   Now use a change of variables in numerator and denominator which takes $B(x, r)$ to $B(0, 1)$.

   (b) Differentiate with respect to r, move the differentiation under the integral, and apply the chain rule.

   (c) Apply the divergence theorem and use that f is harmonic to conclude that $\phi'(r) = 0$.

   (d) Finish the proof by considering the limit $r \to 0$.