Advanced Calculus and Methods of Mathematical Physics

Homework 6

Due via Gradescope, Wednesday, April 1 before 20:00

Note: Assignments marked (*) will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions can be found in the appendix of Kantorovitz' book.

1. *(Kantorovitz, p. 175, Exercise 1.) Let $F(y) = \int_0^1 \mathrm{e}^{x^2 y}\, \mathrm{d}x$.

   (a) Find $F'(0)$.

   (b) For $y \neq 0$, show that F satisfies the differential equation

   $$2y\, F'(y) + F(y) = \mathrm{e}^y \,.$$

2. (Kantorovitz, p. 175, Exercise 1.) Let $b > 0$. For $f \in C([0, b])$, define $F_0(x) = f(x)$ and

   $$F_n(x) = \frac{1}{(n-1)!} \int_0^x (x - y)^{n-1}\, f(y)\, \mathrm{d}y$$

   for $x \in [0, b]$ and $n = 1, 2, \ldots$. Show that $F_n \in C^n([0, b])$ with

   $$F_n^{(k)} = F_{n-k} \quad \text{for } k = 1, \ldots, n \,.$$

   Remark: This relation shows that if J denotes the *integration operator* on $C([0, b])$ defined by

   $$(Jf)(x) = \int_0^x f(y)\, \mathrm{d}y \,,$$

   then

   $$F_n = J^n f \,.$$

3. (Kantorovitz, p. 177, Exercise 5) Let $0 < a < b$ and

   $$F(y) = \int_{a+y}^{b+y} \frac{\mathrm{e}^{xy}}{x}\, \mathrm{d}x \,.$$

   Calculate $F'(y)$ for $y > 0$.

4. *(Adapted from Kantorovitz, Example 4.1.12) Consider the function

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

on the rectangle $I = [0,1] \times [\alpha, 1]$ for $\alpha \in (0,1)$.

(a) Show that

$$\int_0^1 f(x,y)\,\mathrm{d}x = -\frac{1}{1+y^2}$$

for every fixed $y \in [\alpha, 1]$.

Hint: Note that

$$\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} + y\,\frac{\partial}{\partial y}\frac{1}{x^2+y^2}\,.$$

(b) Note that the result from (a) continuously extends to the unit square $I = [0,1]^2$ and conclude that

$$\int_0^1 \int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y = -\frac{\pi}{4}$$

while

$$\int_0^1 \int_0^1 f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \frac{\pi}{4}\,.$$

(c) Does this contradict the theorem on the exchange of partial integration proved in class? Explain!

5. (Kantorovitz, p. 177, Exercise 7) For $n \in \mathbb{N}$, calculate the iterated integral

$$\int_0^\pi \int_0^1 x^{2n-1}\,\cos(x^n y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

6. Show that a set $B \subset \mathbb{R}^n$ has zero content if it has no more than finitely many limit points.

7. Let $B \subset \mathbb{R}^n$ be a bounded set and define the *characteristic function* of B by

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B\,, \\ 0 & \text{if } x \notin B\,. \end{cases}$$

Show that χ_B is Riemann-integrable if and only if B has content.

8. *(Kantorovitz, p. 177, Exercise 8)

(a) Calculate the iterated integral

$$\int_0^1 \int_0^1 \frac{x}{(1+x^2)(1+xy)} \, \mathrm{d}x \, \mathrm{d}y$$

in two different ways, and prove thereby that

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} \, \mathrm{d}x = \frac{\pi \ln 2}{8} \, .$$

(b) Conclude that

$$\int_0^1 \frac{\arctan x}{1+x} \, \mathrm{d}x = \frac{\pi \ln 2}{8} \, .$$