Advanced Calculus and Methods of Mathematical Physics

Homework 7

Due via Gradescope, Tuesday, April 14 before 20:00

Note: Assignments marked (*) will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions can be found in the appendix of Kantorovitz' book. (All Exercises are from Kantorovitz, Section 4.2.15.)

1. *Let $D \subset \mathbb{R}^2$ be the domain bounded by the parabola $x = y^2$ and the line $x = y$. Compute
$$\int_D \sin \frac{\pi x}{y} \, \mathrm{d}S \, .$$

2. Let $D \subset \mathbb{R}^2$ be the annulus with radii $0 < a < b$. Compute
$$\int_D \arctan \frac{y}{x} \, \mathrm{d}S \, .$$

3. Let $D \subset \mathbb{R}^2$ be the domain bounded by the parabola $y = x^2$ and the lines $x = 1$ and $y = 0$. Compute
$$\int_D y \, \mathrm{e}^{x^5} \, \mathrm{d}S \, .$$

4. Let $R > 0$.

   (a) Let $F \in C^1([0, R^2])$ and set $f = F'$. Show that
   $$\int_{B(0,R)} f(x^2 + y^2) \, \mathrm{d}S = \pi \left(F(R^2) - F(0) \right) .$$

   (b) Calculate $\displaystyle\int_{B(0,R)} \cos(x^2 + y^2) \, \mathrm{d}S$.

   (c) Calculate $\displaystyle\int_{B(0,R)} \exp(-x^2 - y^2) \, \mathrm{d}S$.

5. *Let
$$D = \{(x, y) \in \mathbb{R}^2 \colon x, y \geq 0; \sqrt{x} + \sqrt{y} \leq 1\}\,.$$
Compute
$$\int_D (\sqrt{x} + \sqrt{y})^3 \,\mathrm{d}S\,.$$
Hint: use the map
$$(r, \phi) \mapsto (x, y) = r^4\,(\cos^4\phi, \sin^4\phi)\,.$$

6. Let
$$D = \{(x, y) \in \mathbb{R}^2 \colon 1 \leq x^2 - y^2 \leq 9, 0 \leq x \leq 4, y \geq 0\}\,.$$
Compute
$$\int_D x\,y\,\mathrm{e}^{x^2-y^2} \,\mathrm{d}S\,.$$

7. *Let $0 < a < b$ and
$$D = \{x \in \mathbb{R}^3 \colon x_i \geq 0, a \leq \|x\| \leq b\}\,.$$
For $p, q \in \mathbb{R}$ with $q \geq 0$, compute
$$A_i = \int_D x_i^q\,\|x\|^p \,\mathrm{d}x\,.$$

8. Calculate the volume of the domain in $\mathbb{R}^3$ bounded by the surfaces $z = x^2 + y^2$, $y = x^2$, $y = 1$, and $z = 0$.