Advanced Calculus and
Methods of Mathematical Physics

Homework 8

Due via Gradescope, Friday, April 24 before 20:00

Note: Assignments marked (*) will not be graded. Do not turn them in. However, they will be discussed in the tutorial and example solutions can be found in the appendix of Kantorovitz' book.

1. Recall the definition for a line integral of a vector field $F \in C(D, \mathbb{R}^k)$ on a domain $D \subset \mathbb{R}^k$ along a smooth curve $\gamma \in C^1([a,b], D)$,

$$\int_\gamma F \cdot \mathrm{d}x = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t\,.$$

   Prove, by explicit calculation, that this definition is independent of the choice of parametrization of the curve.

2. *(Kantorovitz, Exercises 4.3.15, Problem 1.) Let γ be the helix parameterized by

$$\gamma(t) = (a\cos t, a\sin t, bt)$$

   with $a, b > 0$.

   (a) Find the arc length $s(t)$ of the arc $\{\gamma(\tau)\colon 0 \le \tau \le t\}$.

   (b) Find the length of one turn of the helix.

   (c) Let $F = (-y, x, z)$ be a vector field in $\mathbb{R}^3$. Calculate the line integral

$$\int_\gamma F \cdot \mathrm{d}x$$

   over the turn of the helix $t \in [0, 2\pi]$.

   (d) Calculate the line integral

$$\int_\gamma \frac{1}{\|x\|}\,\mathrm{d}s$$

   over the same turn of the helix.

3. *(Kantorovitz, Exercises 4.3.15, Problem 3.) The curve $\gamma \subset \mathbb{R}^2$ is given in polar coordinates by the C^1 function

$$r = g(\theta)\,, \quad \theta \in [a, b]\,.$$

   (a) Show that the arc length function on γ is given by

$$s(\theta) = \int_a^\theta \sqrt{g'(\tau)^2 + g(\tau)^2}\,\mathrm{d}\tau\,.$$

   (b) For $g(\theta) = 1 - \cos\theta$, $[a, b] = [0, 2\pi]$, find the length of the curve γ.

4. (Kantorovitz, Exercises 4.3.15, Problem 4.) Let γ be the circle centered at the origin with radius r. Calculate

$$\int_\gamma F \cdot \mathrm{d}x$$

   for

   (a) $F(x) = (x_1 - x_2, x_1 + x_2)$,
   (b) $F(x) = \nabla\phi$ with $\phi(x) = \arctan(x_2/x_1)$.

   Explain the difference between case (a) and case (b) in light of the theory.

5. (Kantorovitz, Exercises 4.3.15, Problem 6.) Let γ be the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

   with counter-clockwise orientation. Let

$$F(x, y) = (y^2/(1 + x^2), 2y\arctan x)\,.$$

   Calculate

$$\int_\gamma F \cdot \mathrm{d}x\,.$$

6. *(Kantorovitz, Exercises 4.4.5, Problem 1.) Find the area $S(D)$ of the domain $D \subset \mathbb{R}^2$ with given boundary $\partial D = \gamma$, where γ is the *cardioid* parameterized by

$$\gamma(t) = (1 - \cos t)(\cos t, \sin t)\,, \quad t \in [0, 2\pi].$$

7. *(Kantorovitz, Exercises 4.4.5, Problem 2.) As a above, where γ is the *hypocycloid* satisfying the equation

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

   with $a, b > 0$.

   Hint: find a trigonometric parametrization.

8. (Kantorovitz, Exercises 4.4.5, Problem 3.) As a above, where γ is the curve with parametrization
$$\gamma(t) = (t^2 - 1)(1, t)\,, \quad t \in [-1, 1]\,.$$

9. (From Moskowitz/Paliogiannis, p. 460.) Evaluate the following line integrals along a curve γ, both directly and by using Green's theorem:

   (a) $\displaystyle\int_\gamma (1-x^2)\, y\, \mathrm{d}x + (1+y^2)\, x\, \mathrm{d}y$, where γ is the unit circle in anti-clockwise orientation.

   (b) $\displaystyle\int_\gamma x\, y^2\, \mathrm{d}y - x^2\, y\, \mathrm{d}x$, where γ is the boundary of the annulus $1 \le x^2 + y^2 \le 4$, in standard orientation.

   (c) $\displaystyle\int_\gamma (y - \sin x)\, \mathrm{d}x + \cos x\, \mathrm{d}y$, where γ is the perimeter of the triangle with vertices $(0, 0)$, $(\frac{\pi}{2}, 0)$, and $(\frac{\pi}{2}, 1)$ in counter-clockwise orientation.