The Matrix Inverse

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1 Properties of the Matrix Inverse

Given a matrix $A \in M(n \times n)$, its inverse A^{-1} is the unique matrix with the property

$$AA^{-1} = I = A^{-1}A\,.$$

Recall the following identities:

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

Moreover, if A is invertible then the solution to the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ can be written

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}\,.$$

This observation tells us how we can compute the matrix inverse once we know how to solve linear equations. We begin by writing $\boldsymbol{b}$ as a linear combination of the standard unit vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$,

$$\boldsymbol{b} = b_1\boldsymbol{e}_1 + \cdots + b_n\boldsymbol{e}_n\,.$$

Then

$$\begin{aligned}
\boldsymbol{x} &= A^{-1}\bigl(b_1\boldsymbol{e}_1 + \cdots + b_n\boldsymbol{e}_n\bigr) \\
&= b_1 A^{-1}\boldsymbol{e}_1 + \cdots + b_n A^{-1}\boldsymbol{e}_n \\
&= \begin{pmatrix} | & & | \\ A^{-1}\boldsymbol{e}_1 & \cdots & A^{-1}\boldsymbol{e}_n \\ | & & | \end{pmatrix} \boldsymbol{b}\,.
\end{aligned}$$

Notice that the vectors $\boldsymbol{x}_1 = A^{-1}\boldsymbol{e}_1, \dots \boldsymbol{x}_n = A^{-1}\boldsymbol{e}_n$ are the columns of A^{-1}. At the same time, we see that these vectors are the solutions to the n linear equations

$$A\boldsymbol{x}_1 = \boldsymbol{e}_1\,, \quad \cdots\,, \quad A\boldsymbol{x}_n = \boldsymbol{e}_n\,.$$

To find A^{-1}, we therefore have to simultaneously solve n inhomogeneous linear equations, which is the essence of the following procedure.

2 Computing the inverse

Form the augmented matrix

Write the matrix A to the left, and the identity matrix to the right. For example, when

$$A = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix},$$

write

$$M = \left(\begin{array}{ccc|ccc} 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right).$$

Reduce to row-echelon form

Use elementary row transformations on the augmented matrix to bring the left-hand matrix into row echelon form.

- If you can obtain the identity matrix in the left-hand block, the matrix is invertible and the final right-hand block is A^{-1}.
- If you obtain a row of zeros in the left-hand block, i.e. if $\operatorname{rank} A < n$, then A is not invertible.

Let's work out the example:

$$\left(\begin{array}{ccc|ccc} 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{reorder rows}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right) \xrightarrow{\text{R2}-2\,\text{R1}\to\text{R2}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right) \xrightarrow{\text{R3}-\text{R2}\to\text{R3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \\ 0 & 0 & \frac{1}{2} & 1 & 2 & -1 \end{array}\right) \xrightarrow[2\,\text{R3}\to\text{R3}]{2\,\text{R2}\to\text{R2}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -4 & 2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right) \xrightarrow[\text{R2}+2\,\text{R3}\to\text{R2}]{\text{R1}-\text{R3}\to\text{R1}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right)$$

Check your solution

We see that

$$A^{-1} = \begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix}$$

and it is easy to check that

$$AA^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I\,.$$