Homework 10 Solutions

Unit "strictly determined games":

1 (b).

$$G = \begin{pmatrix} \boxed{6} & \boxed{3}^* \\ 2 & 1^* \end{pmatrix}$$

The top-right entry is a saddle-point,
the game is strictly determined with value 3.

(d)

$$G = \begin{pmatrix} 2 & 0 & -4^* \\ \boxed{3} & \boxed{4} & \boxed{2}^* \\ 0 & -2 & -3^* \end{pmatrix}$$

The right-middle entry is a saddle-point,
the game is strictly determined with value 2.

(f)

$$G = \begin{pmatrix} \boxed{5} & -3^* & 2 \\ 3 & -1^* & \boxed{4} \end{pmatrix}$$

There is no saddle point, the game is not strictly determined.

2 (a)

$$G = \begin{array}{c} \\ \text{one finger} \\ \text{two fingers} \end{array} \begin{array}{c} \begin{array}{cc} \text{one finger} & \text{two fingers} \end{array} \\ \begin{pmatrix} -2^* & 3 \\ \boxed{3}^* & \boxed{4} \end{pmatrix} \end{array}$$

(b) Optimal strategy: Player I will always show two fingers, Player II will always show one finger, with a gain of 3 points per game for Player I.

4(a).

$$G = \begin{array}{r c} & \begin{array}{cc} \text{no witness} & \text{witness} \end{array} \\ \begin{array}{r} \text{plead innocent} \\ \text{plead guilty} \end{array} & \begin{pmatrix} \boxed{0} & -10^{*} \\ -1 & \boxed{-3}^{*} \end{pmatrix} \end{array}$$

(b) The optimal strategy (under a two-player zero-sum game analysis) is to plead guilty – see saddle point in lower right corner.

Unit "non-strictly determined games":

1(b).

$$G = \begin{pmatrix} \boxed{1} & -1^{*} \\ -4^{*} & \boxed{0} \end{pmatrix}$$

has no saddle-point.

Row player's payoff with strategy $r^T = (p, 1-p)$:

$$(p, (1-p)) \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix} = (p - 4(1-p), -p)$$

$$= (5p-4, \underbrace{-p}_{(*)})$$

Optimal strategy requires $5p-4 = -p \Rightarrow 6p = 4 \Rightarrow p = \frac{2}{3}$

Plugging this into (*), we see that the expected payoff for the row player (= value of the game) is $-\frac{2}{3}$.

Row player's payoff when column player has strategy $c = \begin{pmatrix} q \\ 1-q \end{pmatrix}$:

$$\begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} q-(1-q) \\ -4q \end{pmatrix} = \begin{pmatrix} 2q-1 \\ -4q \end{pmatrix}$$

Optimal strategy for column player requires

$$2q-1 = -4q \Rightarrow 6q = 1 \Rightarrow q = \frac{1}{6}$$

Check for consistency: the expected payoff for the row player is $-4q = -4\frac{1}{6} = -\frac{2}{3}$, so it matches the previous result.

(d)

$$G = \begin{pmatrix} -3^{*} & \boxed{2} \\ \boxed{1} & -4^{*} \end{pmatrix}$$ has no saddle-point.

$$(p, (1-p)) \begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix} = (-3p + (1-p), \; 2p - 4(1-p))$$
$$= (-4p+1, \; 6p-4)$$

$$-4p + 1 = 6p - 4 \Rightarrow 10p = 5 \Rightarrow p = \frac{1}{2}$$

So value of the game is $6p - 4 = 6 \cdot \frac{1}{2} - 4 = -1$

$$\begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} -3q + 2(1-q) \\ q - 4(1-q) \end{pmatrix} = \begin{pmatrix} 2-5q \\ 5q-4 \end{pmatrix}$$

$$2 - 5q = 5q - 4 \Rightarrow 10q = 6 \Rightarrow q = \frac{3}{5}$$

Check value of the game: $5q - 4 = 5 \cdot \frac{3}{5} - 4 = -1$ ✓

Unit "reduction by dominance":

2.

$$G = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 5 & 4 \end{pmatrix}$$

row 1 is dominated by row 2 (and row 3)

Optimal strategies:

$$(p,\ 1-p)\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} = \left(4p + 5(1-p),\ 5p + 4(1-p)\right)$$
$$= (5-p,\ 4+p)$$

$$5 - p = 4 + p \implies 1 = 2p \implies p = \tfrac{1}{2}$$

with value of the game $4 + p = 4\frac{1}{2}$

Optimal strategy vector for original game matrix:

$$r^T = \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

By symmetry, the optimal strategy for column player has $q = \frac{1}{2}$,

so
$$c = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

8.

$$G = \begin{pmatrix} 1 & -3 & -4 & 1 \\ 1 & -4 & -1 & 3 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$

1. C4 is dominated by C3
2. C1 is dominated by C2
3. R2 is dominated by R3
4. R1 is dominated by R3

Note: in this problem, you need to look at columns first: only AFTER dominated columns have been removed is it possible to remove R2 !

By symmetry, as before, we find $p = \frac{1}{2}$, $q = \frac{1}{2}$ for the optimal strategies of the reduced game, so the optimal strategy vectors for the full game are

$$r^T = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

with expected payoff of 0 $\Rightarrow$ the game is fair.